Asymptotic Validity of Batch Means Steady-State Confidence Intervals

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Abstract The method of batch means is a widely applied procedure for constructing steady-state confidence intervals. The traditional theoretical support for the method of batch means has rested on the assumption of a functional central limit theorem for the underlying process. We establish here that the method of batch means is valid for Harris recurrent Markov processes whenever the associated process satisfies a simple (non–functional) central limit theorem. This weaker condition for validity of the method of batch means is also shown to hold in the setting of one-dependent regenerative processes.

1 Introduction

Consider a real-valued stochastic process $Y = (Y(t) : t \geq 0)$, with right-continuous paths having left limits, that represents the output of a simulation. Assume that the goal of the simulation is to compute the steady-state mean of Y. More precisely, suppose that Y obeys a law of large numbers (LLN), so that there exists a (deterministic) constant α for which

$$\overline{Y}(t) \triangleq \frac{1}{t}\int_0^t Y(s)\,\mathrm{d}s \Rightarrow \alpha \tag{1}$$

as $t \to \infty$, where $\Rightarrow$ denotes weak convergence (also known as "convergence in distribution"). The quantity α is called the *steady-state mean* of Y, and the problem of computing α is known in the literature as the *steady-state simulation problem*.

The LLN (1) immediately suggests a sampling-based (Monte Carlo) algorithm for computing α. In particular, simulate Y for t units of simulated time, and return $\overline{Y}(t)$ as the estimator for α. As with all numerical procedures, a key issue is the

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C. Alexopoulos et al. (eds.), *Advancing the Frontiers of Simulation: A Festschrift in Honor of George Samuel Fishman*, International Series in Operations Research & Management Science 133, DOI 10.1007/b110059_5,

question of how fast the algorithm converges to the answer. This issue is generally addressed by the central limit theorem (CLT).

In the presence of (1), it is typically the case that Y also enjoys a CLT, so that there exists a further (deterministic) constant σ such that

$$t^{1/2}\left(\overline{Y}(t)-\alpha\right) \Rightarrow \sigma N(0,1) \tag{2}$$

as $t \to \infty$, where $N(0,1)$ is a normal random variable (rv) with mean zero and unit variance. The quantity σ^2 is called the *time-average variance constant* (TAVC) of Y. Because (2) asserts that $\overline{Y}(t)$ has an error of order $t^{-1/2}$, it is evident that the above simulation-based estimator has a slow rate of convergence. (In particular, to add one additional significant figure of accuracy requires increasing the length t of the simulation run by a factor of 100.) As a consequence, it is important to provide the user of such a procedure with some quantitative assessment of accuracy.

Such an accuracy assessment usually comes in the form of a confidence interval. Suppose that the CLT (2) can be generalized to a multivariate CLT of the form

$$t^{1/2}\left(\overline{Y}_1(t)-\alpha,\ldots,\overline{Y}_m(t)-\alpha\right) \Rightarrow \left(\sigma\sqrt{m}N_1(0,1),\ldots,\sigma\sqrt{m}N_m(0,1)\right) \tag{3}$$

as $t \to \infty$, where

$$\overline{Y}_i(t) = \frac{m}{t}\int_{(i-1)t/m}^{it/m} Y(s)\,\mathrm{d}s$$

is the ith *batch mean* for $1 \le i \le m$, and $N_1(0,1),\ldots,N_m(0,1)$ is a collection of m independent and identically distributed (iid) normal rv's with mean zero and unit variance. (Note that (2) is the special case in which $m = 1$, so that (3) is indeed a generalization of (2).) The continuous mapping principle (see, for example, Billingsley 1999, p. 16) then guarantees that if $\sigma^2 > 0$, then

$$\frac{m^{1/2}\left(\dfrac{1}{m}\displaystyle\sum_{i=1}^{m}\overline{Y}_i(t)-\alpha\right)}{\sqrt{\dfrac{1}{m-1}\displaystyle\sum_{i=1}^{m}\left(\overline{Y}_i(t)-\dfrac{1}{m}\displaystyle\sum_{j=1}^{m}\overline{Y}_j(t)\right)^2}} \Rightarrow t_{m-1} \tag{4}$$

as $t \to \infty$, where t_{m-1} is a Student t rv with $m-1$ degrees of freedom. Hence, if one selects z so that $P(-z \le t_{m-1} \le z) = 1-\delta$, then

$$P\left(-z \le \frac{m^{1/2}(\overline{Y}(t)-\alpha)}{s_m(t)} \le z\right) \to 1-\delta$$

as $t \to \infty$, where

$$s_m(t) = \left(\frac{1}{m-1} \sum_{i=1}^{m} \left(\overline{Y}_i(t) - \frac{1}{m} \sum_{j=1}^{m} \overline{Y}_j(t) \right)^2 \right)^{1/2}.$$

In other words,

$$P\left(\alpha \in \left[\overline{Y}(t) - z\frac{s_m(t)}{\sqrt{m}},\ \overline{Y}(t) + z\frac{s_m(t)}{\sqrt{m}} \right] \right) \to 1 - \delta \tag{5}$$

as $t \to \infty$, so that

$$\left[\overline{Y}(t) - z\frac{s_m(t)}{\sqrt{m}},\ \overline{Y}(t) + z\frac{s_m(t)}{\sqrt{m}} \right]$$

is an asymptotic $100(1 - \delta)\%$ confidence interval for α. This confidence interval procedure is, not surprisingly, known as the *method of batch means* and is a widely used method for constructing steady-state confidence intervals; see Fishman (1978) for an early textbook discussion of the method of batch means. The above argument establishes that the method of batch means is asymptotically valid whenever (3) holds with $\sigma^2 > 0$.

This paper is devoted to developing general conditions for the validity of the method of batch means. In Section 2, we review the existing state of the literature, and discuss the validity of the method of batch means under a functional central limit theorem hypothesis. Section 3 shows that when Y shift-couples to a stationary version Y^*, the validity of batch means for Y can be reduced to verifying validity for Y^*. Turning next to validity for Y^*, Theorem 3 of Section 4 proves that the method of batch means holds under a (non-functional) central limit theorem hypothesis, provided that Y^* satisfies a condition closely related to the requirement of ergodicity. Finally, Section 5 applies the theory of Section 3 and 4 to prove the main result of this paper, Theorem 5. Specifically, Theorem 5 proves that when Y is a one-dependent regenerative process, then the method of batch means holds under a simple (non-functional) central limit theorem.

2 Validity Based on a FCLT Hypothesis

Note that the TAVC σ appears in (3), whereas the unknown σ is not present in (4). The reason, of course, is that the common factor σ appears in the limit distribution for both the numerator and denominator of the left-hand side of (4), so that σ is "cancelled out." Steady-state confidence interval procedures that are based on cancellation of the TAVC σ are, not surprisingly, called "cancellation methods." In particular, steady-state confidence intervals based on "standardized time series" (as introduced by Schruben 1983) form a general class of cancellation methods, of which batch means is a special case; see Glynn and Iglehart (1990) for a discussion of the theory of standardized series.

As noted in Schruben (1983) and Glynn and Iglehart (1990), standardized time series steady-state confidence intervals (and hence batch means) are asymptotically valid when Y is presumed to satisfy a functional CLT (FCLT) with $\sigma^2 > 0$.

Definition 1 The process $Y = (Y(t) : t \geq 0)$ is said to satisfy a FCLT if there exist (deterministic) constants α and σ such that

$$\varepsilon^{-1/2} Z_\varepsilon \Rightarrow \sigma B \tag{6}$$

as $\varepsilon \downarrow 0$ (where the weak convergence in (6) is with respect to the Skorohod topology on the function space $D[0, \infty)$ of right-continuous functions with left limits), with Z_ε defined as

$$Z_\varepsilon(t) = \varepsilon \left(\int_0^{t/\varepsilon} Y(s)\,\mathrm{d}s - \alpha t/\varepsilon \right)$$

and where B is standard Brownian motion.

As noted above, the method of batch means provides an asymptotically valid confidence interval (as specified via the limit theorem (5)) whenever Y satisfies a FCLT with $\sigma^2 > 0$. (A direct proof of validity is also straightforward. Note that $g(x) = (x(1/m), x(2/m), \ldots, x(1))$ is a continuous mapping from $D[0, \infty)$ to R^m, so that the continuous mapping principle immediately establishes (3)).

A large class of discrete-event simulations can be represented as Markov process simulations. Specifically, it is typically the case that $Y(t)$ can then be represented as $Y(t) = f(X(t))$, where $X = (X(t) : t \geq 0)$ is Markov and $f : S \to R$ is a given performance measure. In the discrete-event setting, the Markov state $X(t)$ at time t must incorporate both the "physical state" of the system (e.g. the vector number-in-system process) and the state of the future-event schedule (e.g. the remaining time to the next scheduled event for each event type in the system). Under modest conditions on the discrete-event simulation, the Markov process X is positive Harris recurrent, and contains embedded regenerative structure; see Glynn and Haas (2006) for definitions and details. In view of this fact, it is of significant interest to know when a regenerative process satisfies a FCLT. (This easily translates into conditions on the function f appearing in the representation $Y(t) = f(X(t))$.)

Definition 2 Let $Y = (Y(t) : t \geq 0)$ be a real-valued stochastic process. Then, Y is *(classically) regenerative* if there exist random times $0 \leq T(0) < T(1) < \cdots$ such that:

(i) $W_1, W_2, \ldots$ is a sequence of identically distributed random elements; and
(ii) $W_0, W_1, \ldots$ is a sequence of independent random elements,

where $T(-1) = 0$ and for $i \geq 0$,

$$W_i(t) = \begin{cases} Y(T(i-1)+t)\ , & \text{if}\ \ 0 \le t < \tau_i \triangleq T(i) - T(i-1)\ , \\ \Delta\ , & \text{if}\ \ t \ge \tau_i\ , \end{cases} \tag{7}$$

with Δ chosen as a point not in R (for example, we can set $\Delta = (0, 0)$). The regenerative process Y is said to be *positive recurrent* if $E\tau_1 < \infty$.

In the setting of such regenerative processes, we can provide a necessary and sufficient condition for validity of the FCLT; see Theorem 1 of Glynn and Whitt (1993). We assume throughout the remainder of this paper that:

Assumption A $\int_0^t |Y(s)|\,\mathrm{d}s < \infty$ a.s. for each $t \ge 0$.

Theorem 1 *Suppose that* $Y = (Y(t) : t \ge 0)$ *is a real-valued positive recurrent classically regenerative process satisfying Assumption A. Then, there exists* α *and* σ *such that*

$$\varepsilon^{-1/2} Z_\varepsilon \Rightarrow \sigma B$$

as $\varepsilon \downarrow 0$ *(in the sense of weak convergence on* $D[0,\infty)$*) if and only if*

$$E\left(\int_0^{\tau_1} [Y(T(0)+s) - \alpha]\,\mathrm{d}s\right)^2 < \infty \tag{8}$$

and

$$t^2 P\left(\sup_{0\le s\le \tau_1}\left|\int_0^s [Y(T(0)) + u) - \alpha]\ \mathrm{d}u\right| > t\right) \to 0 \quad \text{as}\ t \to \infty\ . \tag{9}$$

It follows that if Y is a positive recurrent classically regenerative process satisfying A, then (3) is valid if and only if (8) and (9) hold, in which case the method of batch means provides asymptotically valid confidence intervals.

3 Validity Based on a CLT Hypothesis: Reduction to the Stationary Setting

In the next two sections, we prove that the method of batch means can be valid even in situations in which the FCLT fails to be satisfied. In particular, we will prove that the method of batch means is asymptotically valid under a CLT hypothesis, plus a modest additional regularity condition.

We establish in this section that the validity of (3) can typically be reduced to the setting in which Y is a stationary stochastic process. Let $\|\cdot\|$ be the *total variation norm* defined by

$$\|P_1 - P_2\| \triangleq \sup\{|P_1(A) - P_2(A)| : A \text{ is measurable}\}\ .$$

Theorem 2 *(i) Suppose that* Y *satisfies Assumption A and that there exists* $Y^* = (Y^*(t) : t \geq 0)$ *for which*

$$\left\| \frac{1}{t} \int_0^t P((Y(u+s) : u \geq 0) \in \cdot\,)\, \mathrm{d}s - P((Y^*(u) : u \geq 0) \in \cdot\,) \right\| \longrightarrow 0 \tag{10}$$

as $t \to \infty$*. Then* Y^* *satisfies Assumption A, so that*

$$\int_0^t |Y^*(s)|\, \mathrm{d}s < \infty \quad a.s.$$

for each $t \geq 0$*.*

(ii) If Y *satisfies Assumption A, (10), and there exists* α *and* σ *such that*

$$t^{1/2} \left(\frac{m}{t} \int_0^{t/m} [Y^*(s) - \alpha]\, \mathrm{d}s, \ldots, \frac{m}{t} \int_{(m-1)t/m}^{t} [Y^*(s) - \alpha]\, \mathrm{d}s \right)$$
$$\Rightarrow (\sigma \sqrt{m} N_1(0,1), \ldots, \sigma \sqrt{m} N_m(0,1)) \tag{11}$$

as $t \to \infty$*, then* $Y = (Y(t) : t \geq 0)$ *satisfies (3).*

Proof For part (i), note that Assumption A guarantees that

$$P\left(\int_0^t |Y(s)|\, \mathrm{d}s < \infty, t \in Q_+ \right) = 1 \tag{12}$$

where Q_+ is the set of nonnegative rational numbers. Since

$$\int_0^t |Y(s)|\, \mathrm{d}s$$

is clearly a nondecreasing function of t, it follows from (12) that

$$P\left(\int_0^t |Y(s)|\, \mathrm{d}s < \infty,\ t \in R_+ \right) = 1\ , \tag{13}$$

where $R_+ \triangleq [0, \infty)$.

We next establish that the limit Y^* appearing in (10) must necessarily be stationary. To see this, we observe that for each $\gamma \geq 0$,

$$\begin{aligned} P((Y^*(u) : u \geq 0) \in \cdot) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t P((Y(u+s) : u \geq 0) \in \cdot\,)\, \mathrm{d}s \\ &= \lim_{t\to\infty} \frac{1}{t+\gamma} \int_0^{t+\gamma} P((Y(u+s) : u \geq 0) \in \cdot\,)\, \mathrm{d}s \end{aligned}$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_{\gamma}^{t+\gamma} P((Y(u+s) : u \geq 0) \in \cdot\,)\, \mathrm{d}s$$
$$= \lim_{t\to\infty} \frac{1}{t} \int_{0}^{t} P((Y(u+s+\gamma) : u \geq 0) \in \cdot\,)\, \mathrm{d}s$$
$$= P((Y^*(u+\gamma) : u \geq 0) \in \cdot\,)\,.$$

Assumption (10) therefore ensures that Y shift-couples to Y^*; see pp. 162 and 167 of Thorisson (2000). In particular, we can presume that Y and Y^* are defined on a common probability space upon which two finite-valued rv's T_1 and T_2 can be defined, and for which

$$Y(T_1+u) = Y^*(T_2+u) \tag{14}$$

for $u \geq 0$. Because Y^* is stationary,

$$P\left(\int_0^t |Y^*(s)|\,\mathrm{d}s > c\right) = P\left(\int_\gamma^{\gamma+t} |Y^*(s)|\,\mathrm{d}s > c\right)$$

for $\gamma \geq 0$. Hence,

$$P\left(\int_0^t |Y^*(s)|\,\mathrm{d}s > c\right) \leq P\left(\int_\gamma^{\gamma+t} |Y^*(s)|\,\mathrm{d}s > c, T_2 \leq \gamma\right) + P(T_2 > \gamma)$$
$$\leq P\left(\int_{T_2}^{T_2+\gamma+t} |Y^*(s)|\,\mathrm{d}s > c\right) + P(T_2 > \gamma)$$
$$= P\left(\int_{T_1}^{T_1+\gamma+t} |Y(s)|\,\mathrm{d}s > c\right) + P(T_2 > \gamma) \quad \text{(due to (14))}$$
$$\leq P\left(\int_{0}^{T_1+\gamma+t} |Y(s)|\,\mathrm{d}s > c\right) + P(T_2 > \gamma)\,.$$

It is therefore evident from (13) that

$$\limsup_{c\to\infty} P\left(\int_0^t |Y^*(s)|\,\mathrm{d}s > c\right) \leq P(T_2 > \gamma)\,.$$

Since $\gamma \geq 0$ was arbitrary, we can now send $\gamma \downarrow 0$ to conclude that

$$\limsup_{c\to\infty} P\left(\int_0^t |Y^*(s)|\,\mathrm{d}s > c\right) = 0\,. \tag{15}$$

So, Assumption A is satisfied by Y^*.

Turning next to part (*ii*), we shall prove (3) when $m = 1$; the proof for general m is essentially identical (but with a higher notational burden). Put $Y_c(t) = Y(t) - \alpha$, $Y_c^*(t) = Y^*(t) - \alpha$, and $a \wedge b \triangleq \min(a, b)$ for $a, b \in R$. Then,

$$
\begin{aligned}
t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s &= t^{-1/2}\int_0^{t\wedge T_1} Y_c(d)\,\mathrm{d}s + t^{-1/2}\int_{T_1}^t Y_c(s)\,\mathrm{d}s\,I(T_1 \le t)\\
&= t^{-1/2}\int_0^{t\wedge T_1} Y_c(d)\,\mathrm{d}s + t^{-1/2}\int_0^{t-T_1} Y_c(T_1+s)\,\mathrm{d}s\,I(T_1 \le t)\\
&= t^{-1/2}\int_0^{t\wedge T_1} Y_c(d)\,\mathrm{d}s + t^{-1/2}\int_0^{t-T_1} Y_c^*(T_2+s)\,\mathrm{d}s\,I(T_1 \le t)\\
&= t^{-1/2}\int_0^{t\wedge T_1} Y_c(d)\,\mathrm{d}s + t^{-1/2}\int_{T_2}^{t+T_2-T_1} Y_c^*(s)\,\mathrm{d}s\,I(T_1 \le t)\\
&= t^{-1/2}\int_0^{t\wedge T_1} Y_c(d)\,\mathrm{d}s + t^{-1/2}I(T_1 \le t)\times\left(\int_0^t Y_c^*(s)\,\mathrm{d}s\right.\\
&\quad \left. - \int_0^{T_2} Y_c^*(s)\,\mathrm{d}s + \int_t^{t+T_2-T_1} Y_c^*(s)\,\mathrm{d}s\right).
\end{aligned}
$$

Clearly,

$$
\left|t^{-1/2}\int_0^{t\wedge T_1} Y_c(s)\,\mathrm{d}s\right| \le t^{-1/2}\int_0^{T_1} |Y_c(s)|\ \mathrm{d}s \Rightarrow 0 \tag{16}
$$

as $t \to \infty$, because (13) implies that $\int_0^{T_1} |Y_c(s)|\,\mathrm{d}s < \infty$ a.s. in view of the finiteness of T_1.

Furthermore, on $\{T_1 \le r, T_2 \le r\}$,

$$
\begin{aligned}
\left|t^{-1/2}\int_0^{t+T_2-T_1} Y_c^*(s)\,\mathrm{d}s\right| &\le t^{-1/2}\int_{t-r}^{t+r} |Y_c^*(u)|\,\mathrm{d}u\\
&\overset{\mathcal{D}}{=} t^{-1/2}\int_0^{2r} |Y_c^*(u)|\,\mathrm{d}u \Rightarrow 0\,,
\end{aligned} \tag{17}
$$

where $\overset{\mathcal{D}}{=}$ denotes "equality in distribution" and (15) was used in the final step. So,

$$
\begin{aligned}
&P\left(t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s \le x\right)\\
&= P\left(t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s \le x, T_1 \le r, T_2 \le r\right) + P(T_1 > r) + P(T_2 > r)\\
&\le P\Bigg(t^{-1/2}\int_0^{t\wedge T_1} Y_c(s)\,\mathrm{d}s + t^{-1/2}\int_0^t Y_c^*(s)\,\mathrm{d}s\,I(T_1 \le t)\\
&\qquad - t^{-1/2}\int_{t-r}^{t+r} |Y_c^*(u)|\,\mathrm{d}u\,I(T_1 \le t) \le x\Bigg) + P(T_1 > r) + P(T_2 > r)\,.
\end{aligned}
$$

Exploiting (11), (16), and (17) yields

$$\limsup_{t\to\infty} P\left(t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s \le x\right) \\ \le P(\sigma N(0,1) \le x) + P(T_1 > r) + P(T_2 > r)\;.$$

Since $r \ge 0$ was arbitrary, we can send $r \downarrow 0$ to conclude that

$$\limsup_{t\to\infty} P\left(t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s \le x\right) \le P(\sigma N(0,1) \le x)\;.$$

A similar argument proves that

$$\liminf_{t\to\infty} P\left(t^{-1/2}\int_0^t Y_c(s)\,\mathrm{d}s \le x\right) \ge P(\sigma N(0,1) \le x)\;,$$

proving part (*ii*) of the theorem for $m = 1$. □

We note that the proof establishes that the limit process Y^* appearing in (10) must necessarily be a stationary process. We conclude this section by showing that in typical discrete-event simulation contexts, we can expect (10) to be a relatively benign hypothesis.

Any one-dependent regenerative process for which $E\tau_1 < \infty$ automatically satisfies (10); see Glynn and Sigman (1992) for details. Since all Markov processes that are positive recurrent in the sense of Harris fall into this class (see Sigman 1990), (10) also holds for this important class of processes. As noted earlier, such Harris recurrence broadly applies to discrete-event simulations; see Glynn and Haas (2006).

4 Validity Based on a CLT Hypothesis: The Main Result

The discussion of Section 3 permits us to presume that we can reduce our analysis to that of a stationary process $Y^* = (Y^*(t) : t \ge 0)$. Without loss of generality, we can extend Y^* to a two-sided version $(Y^*(t) : -\infty < t < \infty)$. Our next result is the main theorem in this paper. It shows that the method of batch means is valid when Y^* satisfies the CLT, plus a modest additional regularity condition (namely (18)).

Theorem 3 *Suppose that Y^* satisfies Assumption A and that*

$$\sup\left\{\left|\frac{1}{t}\int_0^t P\big((Y^*(r) : r \le 0) \in B_1, (Y^*(s+u) : u \ge 0) \in B_2\big)\,\mathrm{d}s \right.\right. \\ \left.\left. - P\left((Y^*(r) : r \le 0) \in B_1\right) P\left((Y^*(u) : u \ge 0) \in B_2\right)\right| : B_1, B_2 \textit{ measurable}\right\} \\ \to 0 \textit{ as } t \to \infty\;. \tag{18}$$

If Y^* satisfies the CLT (2), then Y^* also automatically satisfies (3) for each $m \geq 1$.

Proof To simplify the proof notationally, we specialize to the setting where $m = 2$; the proof for general m is essentially identical. Note that the stationarity of Y^* implies that

$$P\left(t^{-1/2}\int_0^t Y_c^*(s)\,\mathrm{d}s \leq x,\ t^{-1/2}\int_t^{2t} Y_c^*(s)\,\mathrm{d}s \leq y\right) = P\left(t^{-1/2}\int_{-t}^0 Y_c^*(s)\,\mathrm{d}s \leq x,\ t^{-1/2}\int_0^t Y_c^*(s)\,\mathrm{d}s \leq y\right). \tag{19}$$

Furthermore, according to (18), for each $\varepsilon > 0$, there exists r_0 such that

$$\left|\frac{1}{r_0}\int_0^{r_0} P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_s^{s+t} Y_c^*(u)\,\mathrm{d}u \leq y\right)\mathrm{d}s - P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x\right)P\left(t^{-1/2}\int_0^t Y_c^*(u)\,\mathrm{d}u \leq y\right)\right| < \varepsilon \tag{20}$$

uniformly in $t > 0$ and $x, y \in R$.

In addition, the following inequalities hold uniformly in $s \in [0, r_0]$ and $\delta > 0$:

$$\begin{aligned}
&P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_0^t Y_c^*(u)\,\mathrm{d}u \leq y\right)\\
&= P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_s^{s+t} Y_c^*(u)\,\mathrm{d}u + t^{-1/2}\int_0^s Y_c^*(u)\,\mathrm{d}u \right.\\
&\qquad\left. - t^{-1/2}\int_t^{t+s} Y_c^*(u)\,\mathrm{d}u \leq y\right)\\
&\leq P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_s^{s+t} Y_c^*(u)\,\mathrm{d}u \leq y + 2\delta,\right.\\
&\qquad\left. t^{-1/2}\int_0^s Y_c^*(u)\,\mathrm{d}u \leq \delta,\ t^{-1/2}\int_t^{t+s} Y_c^*(u)\,\mathrm{d}u \geq -\delta\right)\\
&\quad + P\left(t^{-1/2}\int_0^s Y_c^*(u)\,\mathrm{d}u > \delta\right) + P\left(t^{-1/2}\int_t^{t+s} Y_c^*(u)\,\mathrm{d}u < -\delta\right)\\
&\leq P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_s^{s+t} Y_c^*(u)\,\mathrm{d}u \leq y + 2\delta\right)\\
&\quad + P\left(t^{-1/2}\int_0^{r_0} \left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) + P\left(t^{-1/2}\int_t^{t+r_0} \left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right)\\
&\leq P\left(t^{-1/2}\int_{-t}^0 Y_c^*(u)\,\mathrm{d}u \leq x,\ t^{-1/2}\int_s^{s+t} Y_c^*(u)\,\mathrm{d}u \leq y + 2\delta\right)\\
&\quad + 2P\left(t^{-1/2}\int_0^{r_0} \left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) \quad \text{(using the stationarity of } Y\text{)}.
\end{aligned} \tag{21}$$

Hence,

$$\begin{aligned}
&P\left(t^{-1/2}\int_{-t}^{0} Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le y\right) \\
&= \frac{1}{r_0}\int_0^{r_0} P\left(t^{-1/2}\int_{-t}^{0} Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le y\right)\mathrm{d}s \\
&\le \frac{1}{r_0}\int_0^{r_0} P\left(t^{-1/2}\int_{-t}^{0} Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_{s}^{s+t} Y_c^*(u)\,\mathrm{d}u \le y+2\delta\right)\mathrm{d}s \\
&\quad +2P\left(t^{-1/2}\int_0^{r_0}\left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) \quad \text{(using (21))} \\
&\le P\left(t^{-1/2}\int_{-t}^{0} Y_c^*(u)\,\mathrm{d}u \le x\right) P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le y+2\delta\right)+\varepsilon \\
&\quad +2P\left(t^{-1/2}\int_0^{r_0}\left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) \quad \text{(using (20))} \\
&= P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le x\right) P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le y+2\delta\right)+\varepsilon \\
&\quad +2P\left(t^{-1/2}\int_0^{r_0}\left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) \quad \text{(using the stationarity of } Y\text{)}\,. \qquad (22)
\end{aligned}$$

It follows from (19), (22), and Assumption A that

$$\begin{aligned}
&\limsup_{t\to\infty} P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_{t}^{2t} Y_c^*(u)\,\mathrm{d}u \le y\right) \\
&\quad\le \limsup_{t\to\infty} P\left(t^{-1/2}\int_{0}^{t} Y_c(u)\,\mathrm{d}u \le x\right) P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le y+2\delta\right) \\
&\qquad +\varepsilon+2P\left(t^{-1/2}\int_0^{r_0}\left|Y_c^*(u)\right|\,\mathrm{d}u > \delta\right) \quad \text{(using (22))} \\
&\quad\le P\left(\sigma N(0,1)\le x\right) P\left(\sigma N(0,1)\le y+2\delta\right)+\varepsilon \quad \text{(using (2))}\,.
\end{aligned}$$

Sending $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$ provides the inequality

$$\begin{aligned}
&\limsup_{t\to\infty} P\left(t^{-1/2}\int_{0}^{t} Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_{t}^{2t} Y_c^*(u)\,\mathrm{d}u \le y\right) \\
&\qquad\qquad \le P\left(\sigma N(0,1)\le x\right) P(\sigma N(0,1)\le y)\,.
\end{aligned}$$

A similar argument establishes that

$$\liminf_{t\to\infty} P\left(t^{-1/2}\int_0^t Y_c^*(u)\,\mathrm{d}u \le x,\ t^{-1/2}\int_t^{2t} Y_c^*(u)\,\mathrm{d}u \le y\right)$$
$$\ge P\left(\sigma N(0,1) \le x\right) P(\sigma N(0,1) \le y)\ ,$$

proving the theorem. □

We note that (18) implies that

$$\sup\left\{\left|\frac{1}{t}\int_0^t P\left((Y^*(r) : r \le 0) \in B_1, (Y^*(s+u) : u \ge 0) \in B_2\right) \mathrm{d}s \right.\right. \tag{23}$$
$$\left.\left. - P\left((Y^*(r) : r \le 0) \in B_1, (Y^*(u) : u \ge 0) \in B_2\right)\right| \ :\ B_2 \text{ measurable}\right\} \to 0$$

as $t \to \infty$, for each fixed (measurable) B_1, so that (18) implies Cesaro mixing; see p. 199 of Thorisson (2000) for the definition (and the remark concerning the need to verify (23) only for finite-dimensional sets B_1). As a consequence of Theorem 2.2 of Chapter 6 of Thorisson (2000), this is equivalent to asserting that the invariant σ-algebra of Y^* is trivial. Of course, since Y^* is stationary, triviality of the invariant σ-algebra is equivalent to ergodicity of Y^*. In other words, (23) (which is a slight weakening of (18)) is equivalent to asserting that Y^* is an ergodic stationary stochastic process.

We further note that the condition (18) is weaker than requiring that Y^* is mixing. All mixing conditions that are present in the literature minimally require that

$$P\left((Y^*(r) : r \le 0) \in B_1, (Y^*(s+u) : u \ge 0) \in B_2\right)$$
$$\to P((Y^*(r) : r \le 0) \in B_1)P((Y^*(u) : u \ge 0) \in B_2) \tag{24}$$

as $s \to \infty$ for each (measurable) B_1, B_2. But (24) precludes stationary stochastic processes that exhibit periodic behavior (e.g. all periodic positive recurrent Harris chains violate (24)). Our formulation of (18) as a (slight) strengthening of Cesaro mixing (which is equivalent to ergodicity) is intended to permit easy application of our results to generic stationary processes, regardless of whether the process is periodic or not.

5 Validity Based on a CLT Hypothesis: Specializing the Results to the Regenerative Setting

As argued in Section 2, the typical discrete-event simulation can be viewed as the simulation of a corresponding Markov process. Under modest conditions on the simulation, the Markov process is positive recurrent in the sense of Harris. Such a Markov process contains embedded regenerative structure under which Y is a one-dependent regenerative process. Specifically, the cycle variables $W_0, W_1, \ldots$ then satisfy Definition 2, part (i); and Definition 2, part (ii) is replaced by:

(ii′) $W_0, W_1, \ldots$ is a one-dependent sequence of random elements (so that $(W_0, W_1, \ldots, W_{i-1})$ is independent of $(W_{i+1}, W_{i+2}, \ldots)$ for $i \geq 1$).

Theorem 4 *Let $Y^* = (Y^*(t) : -\infty < t < \infty)$ be a stationary one-dependent positive recurrent regenerative process. Then (18) is automatically satisfied. Furthermore, if*

$$\int_0^{\tau_1} |Y^*(\tau(0) + s)| \, \mathrm{d}s < \infty \quad \text{a.s.} \, , \tag{25}$$

then Assumption A is satisfied by Y^*.

Proof Let f and g be two nonnegative (measurable) functions that are bounded by one in absolute value. It is a standard fact in the theory of Palm processes that

$$Eg(Y^*(u) : u \geq 0) = \lambda E\Gamma_1(g)$$

where $\lambda = 1/E\tau_1$ and

$$\Gamma_j(g) \triangleq \int_0^{\tau_j} g(Y^*(T(j-1) + s)) \, \mathrm{d}s$$

for $j \geq 1$; see for example, Thorisson (2000).

Furthermore, because the sequence $(W_n : n \geq 0)$ is one-dependent, it follows from the strong law of large numbers for iid sequences that

$$\frac{1}{n} \sum_{j=0}^{n-1} \tau_{2j} \to E\tau_1 \quad \text{a.s.}$$

and

$$\frac{1}{n} \sum_{j=0}^{n-1} \tau_{2j+1} \to E\tau_1 \quad \text{a.s.}$$

as $n \to \infty$ so that

$$\frac{1}{n} \sum_{j=0}^{n-1} \tau_j \to E\tau_1 \quad \text{a.s.}$$

as $n \to \infty$. Hence, if $N(t) = \max\{n \geq -1 : T(n) \leq t\}$, the inequality

$$\frac{T(N(t))}{N(t)} \le t \le \frac{T(N(t)+1)}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

implies that

$$\frac{N(t)}{t} \to \lambda \quad \text{a.s.} \tag{26}$$

as $t \to \infty$.

It follows that for each $\varepsilon > 0$,

$$\begin{aligned}
&Ef(Y^*(r) : r \le 0)\frac{1}{t}\int_0^t g(Y^*(s+u) : u \ge 0)\,\mathrm{d}s \\
&\quad - Ef(Y^*(r) : r \le 0)Eg(Y^*(u) : u \ge 0) \\
&\le Ef(Y^*(r) : r \le 0)\frac{1}{t}\int_0^{T(0)\wedge t} g(Y^*(s+u) : u \ge 0)\,\mathrm{d}s \\
&\quad + Ef(Y^*(r) : r \le 0)\frac{1}{t}\sum_{j=1}^{N(t)+1}\Gamma_j(g) - \lambda Ef(Y^*(r) : r \le 0)E\Gamma_1(g) \\
&\quad \text{(since } g \text{ is nonnegative)} \\
&\le E(T(0)\wedge t)\frac{1}{t} + Ef(Y(r) : r \le 0)\frac{1}{t}\sum_{j=1}^{\lceil(\lambda+\varepsilon)t\rceil}\Gamma_j(g) \\
&\quad - \lambda Ef(Y^*(r) : r \le 0)E\Gamma_1(g) + P(N(t) \ge (\lambda+\varepsilon)t) \\
&\quad \text{(since } f \text{ and } g \text{ are bounded by one)} \\
&\le E(T(0)\wedge t)\frac{1}{t} + Ef(Y^*(r) : r \le 0)\Gamma_1(g)\frac{1}{t} \\
&\quad + \lceil(\lambda+\varepsilon)t\rceil\frac{1}{t}Ef(Y^*(r) : r \le 0)E\Gamma_1(g) \\
&\quad - \lambda Ef(Y^*(r) : r \le 0)E\Gamma_1(g) + P(N(t) \ge (\lambda+\varepsilon)t) \\
&\le E(T(0)\wedge t)\frac{1}{t} + E\tau_1 \cdot \frac{1}{t} \\
&\quad \text{(using the one-dependence and identical distribution property of the cycles)} \\
&\quad + \left(\varepsilon + \frac{1}{t}\right)\lambda Ef(Y^*(r) : r \le 0)E\Gamma_1(g) + P(N(t) \ge (\lambda+\varepsilon)t) \\
&\le E(T(0)\wedge t)\frac{1}{t} + \left(\varepsilon + \frac{1}{t}\right) + P(N(t) \ge (\lambda+\varepsilon)t) + \frac{1}{t}E\tau_1 \\
&\quad \text{(since } f \text{ and } g \text{ are bounded by one).}
\end{aligned}$$

Note that $(T(0) \wedge t)/t$ converges a.s. to zero and is bounded by one, so that the Bounded Convergence Theorem ensures that $t^{-1}E(T(0) \wedge t) \to 0$. Also, (26) guarantees that $P(N(t) \geq (\lambda+\varepsilon)t) \to 0$ as $t \to \infty$. Hence, by sending $t \to \infty$ and then $\varepsilon \downarrow 0$, we conclude that

$$\limsup_{t\to\infty} \sup \Big\{ Ef(Y^*(r) : r \leq 0)\frac{1}{t}\int_0^t g(Y^*(s+u) : u \geq 0)\,\mathrm{d}s \\ - Ef(Y^*(r) : r \leq 0)Eg(Y^*(u) : u \geq 0) : \\ f, g \text{ are (measurable) functions bounded by one} \Big\} \leq 0\ .$$

A similar argument proves that

$$\liminf_{t\to\infty} \inf \Big\{ Ef(Y^*(r) : r \leq 0)\frac{1}{t}\int_0^t g(Y^*(s+u) : u \geq 0)\,\mathrm{d}s \\ - Ef(Y^*(r) : r \leq 0)Eg(Y^*(u) : u \geq 0) : \\ f, g \text{ are (measurable) functions bounded by one} \Big\} \geq 0\ ,$$

thereby proving (18).

To prove Assumption A, note that for any $c \geq 0$,

$$\begin{aligned} P\left(\int_0^{T(0)} |Y^*(s)|\,\mathrm{d}s > c\right) \\ &= \lambda E\int_0^{\tau_1} I\left(\int_s^{\tau_1} |Y^*(T(0)+u)|\,\mathrm{d}u > c\right)\mathrm{d}s \\ &\quad \text{(using Palm theory)} \\ &\leq \lambda E\int_0^{\tau_1} I\left(\int_0^{\tau_1} |Y^*(T(0)+u)|\,\mathrm{d}u > c\right)\mathrm{d}s \\ &= \lambda E\tau_1 I\left(\int_0^{\tau_1} |Y^*(T(0)+u)|\,\mathrm{d}u > c\right)\ . \end{aligned}$$

In light of (25), the Dominated Convergence Theorem therefore proves that

$$\lim_{c\to\infty} P\left(\int_0^{T(0)} |Y^*(s)|\,\mathrm{d}s > c\right) = 0\ ,$$

so that

$$\int_0^{T(0)} |Y^*(s)|\,\mathrm{d}s < \infty \quad \text{a.s.}$$

Since (25) also guarantees that

$$\int_{T(k-1)}^{T(k)} |Y^*(u)|\,\mathrm{d}u < \infty \quad \text{a.s.}\ ,$$

it is evident that

$$\int_0^{T(k)} |Y^*(u)|\, \mathrm{d}u < \infty$$

for $k \geq 0$, proving Assumption A. □

We now focus on the implications of our theory for one-dependent regenerative processes.

Proposition 1 *(i) Let $Y^* = (Y^*(t) : t \geq 0)$ be a stationary one-dependent positive recurrent regenerative process. Then, Y^* satisfies Assumption A if and only if*

$$\int_0^{\tau_1} |Y^*(T(0) + s)|\, \mathrm{d}s < \infty \quad \text{a.s.} \tag{27}$$

(ii) Let $Y = (Y(t) : t \geq 0)$ be a one-dependent positive recurrent regenerative process. Then, Y satisfies Assumption A if and only if

$$\int_0^{T(0)} |Y(s)|\, \mathrm{d}s < \infty \quad \text{a.s.} \tag{28}$$

and

$$\int_0^{\tau_1} |Y(T(0) + s)|\, \mathrm{d}s < \infty\ . \tag{29}$$

Proof We start with (*ii*). If Assumption A is in force, then the proof of Theorem 2 shows that

$$P\left(\int_0^t |Y(s)|\, \mathrm{d}s < \infty \text{ for each } t \geq 0\right) = 1\ .$$

Since the $T(j)$'s are finite-valued, it follows that

$$\int_0^{T(1)} |Y(s)|\, \mathrm{d}s < \infty \quad \text{a.s.}\ ,$$

proving (28) and (29). Conversely, if (29) holds, then

$$\int_{T(j-1)}^{T(j)} |Y(s)|\, \mathrm{d}s < \infty \quad \text{a.s.}$$

for each $j \geq 1$, so that (28) shows that

$$\int_0^{T(n)} |Y(s)|\,\mathrm{d}s < \infty \quad \text{a.s.}$$

for each $n \geq 1$. Since $T(n) \to \infty$ a.s., we may conclude that Assumption A is valid.

As for part (*i*), Theorem 4 shows that (27) implies (28) for the process Y^*. Furthermore,

$$\int_0^{\tau_1} |Y^*(T(0)+s)|\,\mathrm{d}s \overset{\mathcal{D}}{=} \int_0^{\tau_1} |Y(T(0)+s)|\,\mathrm{d}s\ ,$$

so (29) is also validated. Hence, part (*ii*) can be applied to prove the result. □

Implicit in assuming the CLT (2) for the process Y is the presumption that Y is a.s. integrable over finite intervals $[0, t]$. In view of Theorems 2, 3, and 4, we have therefore proved the following result.

Theorem 5 *Let* $Y = (Y(t) : t \geq 0)$ *be a positive recurrent one-dependent regenerative process. Then, (2) holds if and only if (3) is valid.*

Corollary 1 *Suppose that* $Y(t) = f(X(t))$*, where* $X = (X(t) : t \geq 0)$ *is an* S*-valued Markov process that is positive recurrent in the sense of Harris and where* $f : S \to R$*. Then (2) holds for* $Y = (Y(t) : t \geq 0)$ *if and only if (3) is valid.*

Proposition 2 and Corollary 1 show that when $\sigma^2 > 0$, the method of batch means provides asymptotically valid confidence intervals for positive recurrent one-dependent processes and for positive recurrent Harris processes whenever Y satisfies a CLT. In particular, the FCLT is not required to establish validity of the method of batch means.

It has been shown in Glynn and Whitt (2002) that the conditions required for validity of a CLT in the classically regenerative context are strictly weaker than those associated with the FCLT. Specifically, Y satisfies the CLT (2) under Assumption A if and only if (8) holds. So condition (9) is the extra condition required to ensure a FCLT. Examples of processes satisfying (8) but not (9) are easy to construct; see Glynn and Whitt (2002).

Our discussion therefore establishes that the method of batch means is valid under the weakest possible conditions (namely, the assumption of a CLT). In contrast, it can easily be shown that the validity of general standardized time series confidence interval methods (specifically those that look at the maximum of the standardized time series) fundamentally relies on the FCLT assumption. Hence, we may conclude that the method of batch means is the standardized time series confidence interval procedure that is most generally applicable.

Acknowledgments The authors wish to thank the reviewer for the suggestions made to improve the readability of this paper.

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