On the Dynamics of a Finite Buffer Queue Conditioned on the Amount of Loss

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Abstract

This paper is concerned with computing large deviations asymptotics for the loss process in a stylized queueing model that is fed by a Brownian input process. In addition, the dynamics of the queue, conditional on such a large deviation in the loss, is calculated. Finally, the paper computes the quasi-stationary distribution of the system and the corresponding dynamics, conditional on no loss occurring.

1 Introduction

There is a large literature on the dynamics of infinite buffer queues conditioned on either large customer delays or a large number-in-system; see, for example, [1,2,6,11]. This paper, on the other hand, makes a contribution to the rare-event literature on finite buffer queues, conditioned on the amount of loss. Our vehicle for studying this problem is two-sided reflected Brownian motion. It is known that this process can be viewed as a heavy-traffic approximation to a finite-buffer system; see, for example, [4]. In this heavy traffic setting, the loss process is then approximated by the local time at the upper boundary b associated with a full buffer. In addition to the intrinsic interest in this specific stylized queueing-type model, we expect the qualitative behavior to be representative of the heavy-traffic rare-event behavior of more general single buffer systems.

To make our contribution more precise, let $X = (X(t) : t \geq 0)$ be the Markov process defined via the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \mu \mathrm{d}t + \sigma \mathrm{d}B(t) + \mathrm{d}L(t) - \mathrm{d}U(t),$$

where $B = (B(t) : t \geq 0)$ is a standard (one-dimensional) Brownian motion and L, U are the minimal continuous non-decreasing processes satisfying $L(0) = U(0) = 0$ for which $X(t) \in [0, b]$ for $t \geq 0$ and

$$\int_{[0,\infty)} I(X(t) > 0)\, \mathrm{d}L(t) = 0$$

and

$$\int_{[0,\infty)} I(X(t) < b)\, \mathrm{d}U(t) = 0.$$

The processes L and U are then called the local time processes at the boundaries 0 and b, respectively; our interest is in the process U. The random variable (rv) $U(t)$ is then the Brownian analog to the cumulative amount of loss over $[0, t]$ and it can be viewed as an approximation to the cumulative loss in a single-server finite buffer queue in heavy traffic.

In Section 2, we compute the typical behavior of U, recovering results due to [3, 17]. Our martingale approach leads to a single differential equation (plus an unknown constant) that is the analog to the Poisson equation that arises in the analysis of additive functionals of the form $\int_0^t f(X(s))\mathrm{d}s$. In contrast, the previous calculations relied on regenerative ideas [17] and the Kella-Whitt martingale [3].

Section 3 turns to the analysis of the rare-event behavior of X, conditioned on $U(t) > \gamma t$ (where $\gamma > r$ and r is the mean rate at which U increases) and $U(t) < \gamma t$ (for $\gamma < r$), when t is large. In other words, we consider the conditional behavior in both the case where the loss is unusually large ($U(t) > \gamma t$, where $\gamma > r$) or unusually small ($U(t) < \gamma t$, where $\gamma < r$). Section 4 develops the dynamics of X in the extreme setting where there is no loss at all. In particular, we compute the quasi-stationary dynamics of X associated with conditioning on $U(t) = 0$ (so that $\tau_b > t$, where τ_b is the first hitting time of b by X) for t large. The calculations of Section 3 and 4 rely on our ability to explicitly compute the solutions to certain eigenvalue problems, and to apply Girsanov's formula as a mechanism for determining the modified drift under the conditioning. Section 5 collects the proofs that involve non-trivial calculations related to explicit computation of the asymptotics and dynamics considered in this paper.

2 The Typical Behavior of U

The typical behavior of U is captured through a central limit theorem (CLT) of the form

$$t^{-\frac{1}{2}}(U(t) - rt) \Rightarrow \eta\mathcal{N}(0,1)$$

as $t \to \infty$, for appropriately chosen constants r and η^2 (where $\Rightarrow$ denotes weak convergence and $\mathcal{N}(0,1)$ is a normal random variable (rv) with mean 0 and unit variance). To compute r and η^2, we will apply the martingale CLT. To write $U(t) - rt$ in terms of a martingale, note that because L and U have no jumps, we can apply Itô's formula to establish that if h is twice differentiable on $[0, b]$, then

$$\mathrm{d}h(X(t)) = (\mathcal{L}h)(X(t))\mathrm{d}t + h'(0)\mathrm{d}L(t) - h'(b)\mathrm{d}U(t) + h'(X(t))\sigma\mathrm{d}B(t), \tag{2.1}$$

where

$$\mathcal{L} = \mu\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Here, we used the fact that because L and U increase only when X takes on the values 0 and b respectively, it follows that $h'(X(t))\mathrm{d}L(t) = h'(0)\mathrm{d}L(t)$ and $h'(X(t))\mathrm{d}U(t) = h'(b)\mathrm{d}U(t)$.

If we choose h so that $(\mathcal{L}h)(x) = r$ on $[0, b]$ subject to $h'(0) = 0$ and $h'(b) = 1$, then

$$U(t) - rt + h(X(t)) - h(X(0)) = \sigma \int_0^t h'(X(s))\,\mathrm{d}B(s),$$

so that $M(t) = U(t) - rt + h(X(t)) - h(X(0))$ is a zero-mean square integrable martingale (adapted to $(\mathcal{F}_t : t \geq 0)$, where $\mathcal{F}_t = \sigma(X(u) : 0 \leq u \leq t)$). This differential equation and its associated boundary conditions determine h only up to an additive constant. We can therefore make h unique by requiring $h(0) = 0$. We are therefore led to the differential equation

$$\begin{aligned} &(\mathcal{L}h)(x) = r, \quad 0 \leq x \leq b \\ \text{s.t.}\quad &h(0) = 0 \\ &h'(0) = 0 \\ &h'(b) = 1 \qquad . \end{aligned} \tag{2.2}$$

One now needs to solve (2.2) for h and r; (2.2) is the Poisson's equation for the local time U that is the analog to the Poisson's equation for additive functionals of the form $\int_0^t f(X(s))\mathrm{d}s$ that has appeared previously in the literature; see, for example, [9]. In any case, the solution (h, r) to (2.2) is

$$r = \begin{cases} \dfrac{\mu}{1-e^{-\rho b}}, & \text{if } \mu \neq 0 \\ \dfrac{\sigma^2}{2b}, & \text{if } \mu = 0 \end{cases} \tag{2.3}$$

and

$$h(x) = \begin{cases} \dfrac{x + \rho\left(e^{-\rho x} - 1\right)}{1-e^{-\rho b}}, & \text{if } \mu \neq 0 \\ \dfrac{x^2}{2b}, & \text{if } \mu = 0 \end{cases}$$

where $\rho = \frac{2\mu}{\sigma^2}$.

To compute η^2, we exploit the martingale CLT; see p.338-340 of [8]. Note that $M(\cdot)$ is a continuous path martingale for which

$$\frac{1}{t}[M](t) = \frac{1}{t}\int_0^t \sigma^2 h'(X(s))^2\,\mathrm{d}s \to \sigma^2 \int_0^b h'(x)^2 \pi(\mathrm{d}x) \triangleq \eta^2$$

a.s. as $t \to \infty$, where π is the stationary distribution of X. The distribution π is given by

$$\pi(\mathrm{d}x) = \begin{cases} \dfrac{\rho e^{\rho x}}{e^{\rho b} - 1}\mathrm{d}x, & \text{if } \mu \neq 0 \\ b^{-1}\mathrm{d}x, & \text{if } \mu = 0 \end{cases} \tag{2.4}$$

for $x \geq 0$; see p.90 of [12]. Upon noting that $t^{-\frac{1}{2}}(h(X(t)) - h(X(0))) \to 0$ a.s. as $t \to 0$, the martingale CLT yields the proposition below.

Proposition 1 *The loss process $U = (U(t) : t \geq 0)$ satisfies the CLT*

$$t^{-\frac{1}{2}}(U(t) - rt) \Rightarrow \eta\mathcal{N}(0,1)$$

as $t \to \infty$, where r is given in (2.3) and

$$\eta^2 = \begin{cases} \dfrac{\sigma^2 e^{2\rho b}\left(e^{\rho b} - 2\rho b - e^{-\rho b}\right)}{\left(e^{\rho b}-1\right)^3}, & \text{if } \mu \neq 0 \\ \dfrac{\sigma^2}{3}, & \text{if } \mu = 0. \end{cases}$$

It follows that for t large, the rv $U(t)$ can be approximated as $U(t) \overset{\mathcal{D}}{\approx} rt + \eta t^{\frac{1}{2}}\mathcal{N}(0,1)$, where $\overset{\mathcal{D}}{\approx}$ means "has approximately the same distribution as" (and carries no rigorous meaning per se).

As noted earlier, the above martingale argument recovers the CLT derived by [17] using more complicated regenerative methods. In the next section, we study the "rare-event" large deviations behavior of the loss process U.

3 Conditional Limits Based on Unusually Large and Small Amounts of Loss

Not surprisingly, the conditional limit behavior of X, given $U(t) > \gamma t$, is linked to the computation of the large deviations probability for the event $\{U(t) > \gamma t\}$ for t large. The Gärtner-Ellis theorem (see, for example, [5]) provides one mechanism for computing such a large deviations probability. In particular, the computation of

$$\lim_{t\to\infty} \frac{1}{t} \log \mathrm{E} \exp(\theta U(t)) \tag{3.1}$$

plays a key role in the calculation. To this end, we attempt to construct a martingale of the form

$$M(t) = \exp(\theta U(t) - \psi t + h(X(t))). \tag{3.2}$$

Of course, ψ and $h(\cdot)$ clearly depend on the choice of θ, but we choose (temporarily) to suppress the dependence on θ in order to simplify our notation.

Applications of Itô's formula and (2.1) establish that

$$\begin{aligned} \mathrm{d}M(t) &= M(t)[\theta \mathrm{d}U(t) - \psi \mathrm{d}t + \mathrm{d}h(X(t))] + \frac{M(t)}{2}[\theta \mathrm{d}U(t) - \psi \mathrm{d}t + \mathrm{d}h(X(t))]^2 \\ &= M(t)[\theta \mathrm{d}U(t) - \psi \mathrm{d}t + h'(0)\mathrm{d}L(t) - h'(b)\mathrm{d}U(t) + (\mathcal{L}h)(X(t))\mathrm{d}t \\ &\quad + h'(X(t))\sigma \mathrm{d}B(t)] + \frac{M(t)}{2} h'(X(t))^2 \sigma^2 \mathrm{d}t \\ &= M(t)[(\mathcal{L}h)(X(t)) - \psi + \frac{\sigma^2}{2} h'(X(t))^2]\mathrm{d}t + M(t)h'(0)\mathrm{d}L(t) \\ &\quad + M(t)(\theta - h'(b))\mathrm{d}U(t) + M(t)h'(X(t))\sigma \mathrm{d}B(t). \end{aligned} \tag{3.3}$$

In order that M be a martingale, we should therefore choose h and ψ so that

$$(\mathcal{L}h)(x) + \frac{\sigma^2}{2} h'(x)^2 = \psi, \tag{3.4}$$

subject to $h'(0) = 0$ and $h'(b) = \theta$. Since (3.4) determines h only up to an additive constant, we may add on the boundary condition $h(0) = 0$ in order to uniquely specify h. As an alternative to solving the non-linear differential equation (3.4), we may seek to instead compute $v(x) = \exp(h(x))$. With this change of variables, we find that v is a positive solution of the linear differential equation

$$(\mathcal{L}v)(x) = \psi v(x) \tag{3.5}$$

for $0 \le x \le b$, subject to $v'(0) = 0$, $v'(b) - \theta v(b) = 0$ and $v(0) = 1$. In other words, $v(\cdot) = v(\theta, \cdot)$ is the solution of an eigenvalue problem, and $\psi = \psi(\theta)$ is the corresponding eigenvalue associated with parameter θ.

Assuming that we can find a solution to the eigenvalue problem (3.5), note that the associated h satisfies

$$\begin{aligned} \mathrm{d}h(X(t)) &= (\mathcal{L}h)(X(t))\mathrm{d}t + h'(0)\mathrm{d}L(t) - h'(b)\mathrm{d}U(t) + h'(X(t))\sigma \mathrm{d}B(t) \\ &= (\psi - \frac{\sigma^2}{2} h'(X(t))^2)\mathrm{d}t - \theta \mathrm{d}U(t) + h'(X(t))\sigma \mathrm{d}B(t), \end{aligned}$$

where we used (3.4) for the second equality. Hence,

$$h(X(t)) - h(X(0)) = \psi t - \theta U(t) + \int_0^t h'(X(s))\sigma \, \mathrm{d}B(s) - \frac{1}{2}\int_0^t h'(X(s))^2\sigma^2 \, \mathrm{d}s$$

so that $M(t)$ can then be written as

$$M(t) = \exp\left(\int_0^t h'(X(s))\sigma \, \mathrm{d}B(s) - \frac{1}{2}\int_0^t h'(X(s))^2\sigma^2 \, \mathrm{d}s\right).$$

It is then a standard fact that $(M(t) : t \geq 0)$ is a local martingale adapted to $(\mathcal{F}_t : t \geq 0)$. Furthermore, because the solution h to (3.4) necessarily has a continuous first derivative (since h'' is assumed to exist), which is therefore bounded on $[0, b]$, it is evident that Novikov's condition is satisfied, so that $(M(t) : t \geq 0)$ is a (true) martingale. In view of the boundedness of h over $[0, b]$,

$$\frac{1}{t}\log \mathrm{E}\exp(\theta U(t)) \to \psi$$

as $t \to \infty$. The eigenvalue $\psi = \psi(\theta)$ is therefore precisely the desired limit (3.1).

Given the clear importance of $\psi = \psi(\theta)$ and $v(\cdot) = v(\theta, \cdot)$, we now present the solution to (3.5). In preparation for starting our result, we define the following regions of the parameter space involving θ, μ, and b:

$$\begin{aligned}
\mathscr{R}_1 &= \{(\theta, \mu, b) : \theta > 0\} \\
\mathscr{R}_2 &= \{(\theta, \mu, b) : \theta < 0, \mu(\mu + \theta\sigma^2) \leq 0\} \\
\mathscr{R}_3 &= \{(\theta, \mu, b) : \theta < 0, \mu(\mu + \theta\sigma^2) > 0, b\mu(\mu + \theta\sigma^2) > -\theta\sigma^4\} \\
\mathscr{R}_4 &= \{(\theta, \mu, b) : \theta < 0, \mu(\mu + \theta\sigma^2) > 0, b\mu(\mu + \theta\sigma^2) < -\theta\sigma^4\} \\
\mathscr{B}_1 &= \{(\theta, \mu, b) : \theta = 0\} \\
\mathscr{B}_2 &= \{(\theta, \mu, b) : \theta < 0, \mu(\mu + \theta\sigma^2) > 0, b\mu(\mu + \theta\sigma^2) = -\theta\sigma^4\}
\end{aligned}$$

Theorem 1 *The solutions $\psi = \psi(\theta)$ and $v(\cdot) = v(\theta, \cdot)$ to (3.5) are:*

a.) For $(\theta, \mu, b) \in \mathscr{R}_i (i = 1, 3)$, $\psi(\theta) = \frac{\beta(\theta)^2 - \mu^2}{2\sigma^2}$ and

$$v(\theta, x) = \frac{1}{2\beta(\theta)} e^{-\frac{\mu}{\sigma^2}x}\left[(\beta(\theta) - \mu)e^{-\frac{\beta(\theta)}{\sigma^2}x} + (\beta(\theta) + \mu)e^{\frac{\beta(\theta)}{\sigma^2}x}\right],$$

where $\beta(\theta)$ is the unique root in $\mathscr{I}_i$ of the equation

$$\frac{1}{\beta}\log\left(\frac{(\beta - \mu)(\beta + \mu + \theta\sigma^2)}{(\beta + \mu)(\beta - \mu - \theta\sigma^2)}\right) = \frac{2b}{\sigma^2}, \tag{3.6}$$

with $\mathscr{I}_1 = (|\mu| \vee |\mu + \theta\sigma^2|, \infty)$ and $\mathscr{I}_3 = (0, |\mu| \wedge |\mu + \theta\sigma^2|)$.

b.) For $(\theta, \mu, b) \in \mathscr{R}_i (i = 2, 4)$, $\psi(\theta) = -\frac{\xi(\theta)^2 + \mu^2}{2\sigma^2}$ and and

$$v(\theta, x) = e^{-\frac{\mu}{\sigma^2}x}\left[\cos\left(\frac{\xi(\theta)x}{\sigma^2}\right) + \frac{\mu}{\xi(\theta)}\sin\left(\frac{\xi(\theta)x}{\sigma^2}\right)\right];$$

where $\xi(\theta)$ is the unique root in $(0, \frac{\pi\sigma^2}{b})$ of the equation

$$\frac{b}{\sigma^2}\xi = \arccos\left(\frac{\xi^2 + \mu(\mu + \theta\sigma^2)}{\sqrt{(\xi^2 + \mu(\mu + \theta\sigma^2))^2 + \xi^2\theta^2\sigma^4}}\right) \tag{3.7}$$

c.) For $(\theta, \mu, b) \in \mathscr{B}_1$, $\psi(\theta) = 0$ and $v(\theta, x) \equiv 1$.

d.) For $(\theta, \mu, b) \in \mathscr{B}_2$, $\psi(\theta) = -\frac{\mu^2}{2\sigma^2}$ and

$$v(\theta, x) = e^{-\frac{\mu}{\sigma^2}x} \left(\frac{\mu}{\sigma^2}x + 1\right).$$

The proof of Theorem 1 can be found in Section 5. The Gärtner-Ellis theorem then implies the following result (in which we adopt the standard notation that $\mathrm{P}_x(\cdot) \triangleq \mathrm{P}(\cdot|X(0) = x)$ and $\mathrm{E}_x(\cdot) \triangleq \mathrm{E}(\cdot|X(0) = x)$)..

Theorem 2 *For $\gamma > r$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathrm{P}_x(U(t) > \gamma t) = -I(\gamma)$$

whereas for $0 < \gamma < r$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathrm{P}_x(U(t) < \gamma t) = -I(\gamma),$$

where

$$I(\gamma) = \theta_\gamma \gamma - \psi(\theta_\gamma)$$

and $\psi'(\theta_\gamma) = \gamma$.

Note that U is regenerative with respect to a cycle structure in which $X = 0$ at the regeneration times but hits level b at some time within each cycle. More precisely, U is regenerative with respect to $(\tau_n : n \geq 0)$, where $\tau_0 = 0$ and

$$\tau_n = \inf\{t > \tau_{n-1} : X(t) = 0, \sup_{\tau_{n-1} \leq s < t} X(s) = b\}$$

for $n \geq 1$. It is a well known fact that if $X(0) = 0$, then $U(\tau_1)$ is exponentially distributed with mean $\frac{1-e^{-2\mu b}}{2\mu}$ for $\mu \neq 0$ and b for $\mu = 0$; see, for example, [17]. A curious feature of Theorems 1 and 2 is that $\mathrm{E}_x \exp(\theta U(t)) < \infty$ for $t \geq 0$, while the moment generating function $\mathrm{E}_0 \exp(\theta U(\tau_1))$ of the loss over a typical cycle diverges for $\theta \geq \frac{2\mu}{1-e^{-2\mu b}}$ if $\mu \neq 0$ and for $\theta \geq b^{-1}$ if $\mu = 0$. Evidently, the randomization of the time horizon associated with τ_1 induces heavier tails in the loss process.

To compute the conditional dynamics of X given $\{U(t) > \gamma t\}$, we define $\mathrm{P}_x^\gamma(\cdot)$ so that for each $t \geq 0$,

$$\mathrm{P}_x^\gamma(A) = \mathrm{E}_x \left\{ I(A) \exp\left(\sigma \int_0^t h'(\theta_\gamma, X(s))\, \mathrm{d}B(s) - \frac{\sigma^2}{2} \int_0^t h'(\theta_\gamma, X(s))^2\, \mathrm{d}s\right)\right\}$$

for $A \in \mathcal{F}_t$. By Girsanov's formula,

$$\tilde{B}(t) \triangleq B(t) - \sigma \int_0^t h'(\theta_\gamma, X(s))\, \mathrm{d}s$$

is a standard Brownian motion under P_x^γ, so that X satisfies the SDE

$$\mathrm{d}X(t) = (\mu + \sigma^2 h'(\theta_\gamma, X(t)))\mathrm{d}t + \sigma \mathrm{d}\tilde{B}(t) + \mathrm{d}L(t) - \mathrm{d}U(t),$$

subject to $X(0) = x$, under P_x^γ. In other words, the law of X under P_x^γ is identical to that of the process $X_\gamma = (X_\gamma(t) : t \geq 0)$ satisfying the SDE

$$\mathrm{d}X_\gamma(t) = (\mu + \sigma^2 h'(\theta_\gamma, X_\gamma(t)))\mathrm{d}t + \sigma \mathrm{d}B(t) + \mathrm{d}L_\gamma(t) - \mathrm{d}U_\gamma(t), \tag{3.8}$$

where $L_\gamma(\cdot)$ and $U_\gamma(\cdot)$ are defined analogously for X_γ as in Section 1. We will show that when t is large, X, when conditioned on $\{U(t) > \gamma t\}$, follows the law of X_γ; a similar result holds when conditioning on $\{U(t) < \gamma t\}$ for $\gamma < r$.

We start by noting that for each $\gamma > 0$, X_γ is a positive recurrent Markov process. In particular, note that if τ_0 is the first hitting time of the origin, the stochastic monotonicity of X_γ implies that

$$\begin{aligned}
\mathrm{P}^\gamma_x(\tau_0 \le \Delta) &\ge \mathrm{P}^\gamma_b(\tau_0 \le \Delta) \\
&= \mathrm{E}_b \left\{ I(\tau_0 \le \Delta) \exp\left(\theta_\gamma U(\Delta) - \psi(\theta_\gamma)\Delta + h(\theta_\gamma, X(\Delta)) - h(\theta_\gamma, b)\right)\right\} \\
&\ge \mathrm{P}_b(\tau_0 \le \Delta) \exp\left(-\psi(\theta_\gamma)\Delta + \inf_{0\le x\le b} h(\theta_\gamma, x) - h(\theta_\gamma, b)\right) \\
&> 0
\end{aligned}$$

so $\inf_{0\le x\le b} \mathrm{P}^\gamma_x(\tau_0 \le \Delta) > 0$. Hence, for each $\gamma > 0$, X_γ is uniformly recurrent and hence has a unique stationary distribution $\pi_\gamma(\cdot)$ to which X_γ converges exponentially fast (uniformly in x); see, for example, [15].

A careful justification for the conditional dynamics described above observes that if $f(X_\gamma(u) : 0 \le u \le s)$ is a bounded $\mathcal{F}_s$-measurable rv, then

$$\begin{aligned}
&\mathrm{E}_x[f(X(u) : 0 \le u \le s) | U(t) > \gamma t] \\
=&\frac{\mathrm{E}_x f(X_\gamma(u) : 0 \le u \le s) \exp[-\theta_\gamma(U_\gamma(t) - \gamma t) + h(\theta_\gamma, X(t)) - h(\theta_\gamma, x)] I(U_\gamma(t) > \gamma t)}{\mathrm{E}_x \exp[-\theta_\gamma(U_\gamma(t) - \gamma t) + h(\theta_\gamma, X(t)) - h(\theta_\gamma, x)] I(U_\gamma(t) > \gamma t)}
\end{aligned} \tag{3.9}$$

Note that the denominator of (3.9) takes the same form as the numerator, with $f \equiv 1$. The numerator of (3.9) can be expressed as

$$\int_0^\infty \theta_\gamma e^{-\theta_\gamma y} \mathrm{E}_x f(X_\gamma(u) : 0 \le u \le s) I(0 < U_\gamma(t) - \gamma t < y) \exp(h(\theta_\gamma, X_\gamma(t)) - h(\theta_\gamma, x)). \tag{3.10}$$

We claim that the integral (3.10) is asymptotic to

$$\frac{1}{\theta_\gamma \sqrt{2\pi\psi''(\theta_\gamma)t}} \mathrm{E}_x f(X_\gamma(u) : 0 \le u \le s) \int_0^b e^{h(\theta_\gamma, y)} \pi_\gamma(\mathrm{d}y) \cdot e^{-h(\theta_\gamma, x)} \tag{3.11}$$

The proof of this claim follows an argument similar to that used by [13,14], and hence is omitted. Note that the key to proving (3.10) is a suitable local CLT for $U_\gamma(t) - \gamma t$. Such a local CLT takes advantage of the fact that X_γ is a positive recurrent regenerative process (with regeneration times given, for example, by the times at which X_γ visits 0 having visited b at some intermediate time). Furthermore, if τ is the associated regeneration time, $U_\gamma(\tau) - \gamma\tau$ has a density, since it is the convolution of two independent rv's, one of which is $-\gamma$ times the first passage time from 0 to b of X_γ (which has a density, since the first passage time of X from 0 to b is known to have a density). As a consequence, $U_\gamma(\tau) - \gamma\tau$ has the requisite non-lattice property needed for a local CLT.

By applying (3.10) and (3.11) first with with $f \equiv g$ and second with $f \equiv 1$, we arrive at the conclusion that

$$\mathrm{E}_x[g(X(u) : 0 \le u \le t) | U(t) > \gamma t] \sim \mathrm{E}_x g(X_\gamma(u) : 0 \le u \le t)$$

as $t \to \infty$. We summarize our discussion with Theorem 3.

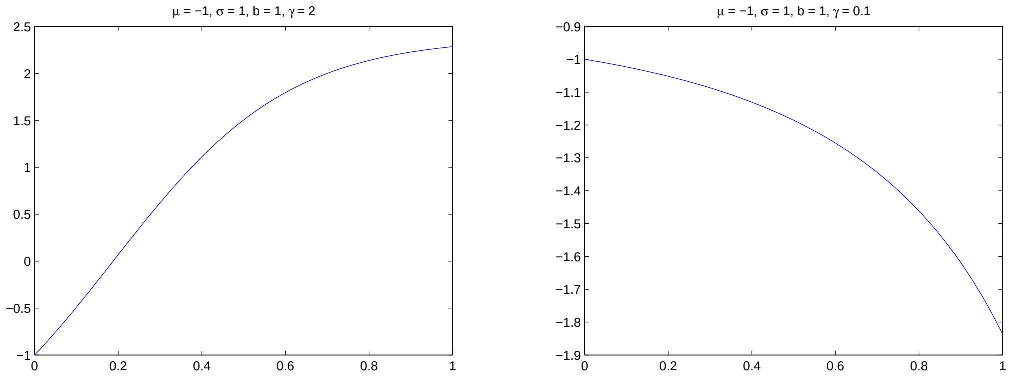

Figure 1: Drift function of X_γ when $\mu = -1$, $\sigma = 1$, and $\gamma = 2$ (Left) or $\gamma = 0.1$ (Right)

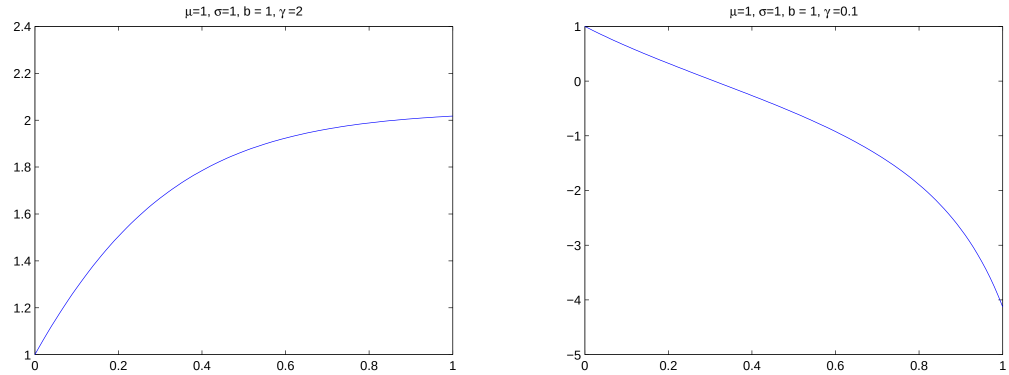

Figure 2: Drift function of X_γ when $\mu = 1$, $\sigma = 1$ and $\gamma = 2$ (Left) or $\gamma = 0.1$ (Right).

Theorem 3 *a.) For $\gamma > r$,*

$$\mathrm{P}_x(U(t) > \gamma t) \sim \frac{1}{\theta_\gamma \sqrt{2\pi\psi''(\theta_\gamma)t}} \exp(-I(\gamma)t) \int_0^b e^{h(\theta_\gamma,y)} \pi_\gamma(\mathrm{d}y) \cdot e^{-h(\theta_\gamma,x)}$$

as $t \to \infty$, whereas for $\gamma < r$,

$$\mathrm{P}_x(U(t) < \gamma t) \sim \frac{1}{\theta_\gamma \sqrt{2\pi\psi''(\theta_\gamma)t}} \exp(-I(\gamma)t) \int_0^b e^{h(\theta_\gamma,y)} \pi_\gamma(\mathrm{d}y) \cdot e^{-h(\theta_\gamma,x)}$$

as $t \to \infty$.

b.) Conditional on $\{U(t) > \gamma t\}$ with $\gamma > r$ (or $\{U(t) < \gamma t\}$ with $\gamma < r$),

$$(X(u) : u \geq 0) \Rightarrow (X_\gamma(u) : u \geq 0)$$

in $C[0, \infty)$ as $t \to \infty$, where X_γ satisfies the SDE (3.8).

4 The Quasi-stationary Distribution for Reflected Brownian Motion

In this section, we focus on the extreme case in which the system has experienced no loss over the interval $[0,t]$. Of course, the strong Markov property implies that if $\tau_b < t$, then $U(t) > 0$ a.s., so conditioning on no loss is equivalent to requiring that $\tau_b > t$. The problem of computing the conditional behavior of X, conditioned on $\tau_b > t$ for t large, is exactly the problem of calculating the associated quasi-stationary distribution for X. This quasi-stationary distribution is identical to that associated with a one-sided reflected Brownian motion exhibiting reflection only at the origin, conditioned on not exceeding level b over $[0,t]$. From a queueing standpoint, we can view this conditioning as one involving an infinite buffer Brownian queue in which the buffer context process has not yet exceeded b by time t. Thus, this quasi-stationary distribution has two queueing interpretations, one in terms of a finite buffer queue and the other in terms of an infinite buffer queue.

To calculate the quasi-stationary behavior, we seek a positive martingale that lives on $[0,b)$, thereby inducing a change-of-measure for X that does not visit b; see, for example, [10]. In particular, we consider a martingale of the form

$$M(t) = \frac{e^{\lambda t} v(X(t))}{v(X(0))}$$

where $v(\cdot)$ is positive on $[0,b)$ and $v(b) = 0$. Using Itô's formula as in Section 3, we find that the pair (v,λ) should satisfy the eigenvalue problem,

$$\mathcal{L}v = -\lambda v, \tag{4.1}$$

subject to $v'(0) = 0$ and $v(b) = 0$. Again, v is only determined up to a multiplicative constant, so we further require that $v(0) = 1$. The spectrum associated with the above eigenvalue problem is continuous (i.e. the set of λ's satisfying (4.1) is a continuum). One way to identify the appropriate λ is on the basis of the fact that we are seeking an associated positive eigenfunction v. However, determining the eigenfunction/eigenvalue pair subject to such a positivity constraint is challenging. We therefore proceed via an alternative (somewhat heuristic) route that we later rigorously verify in Theorem 4.

Note that if (4.1) has a solution v that is positive on $[0,b)$, $(M(t) : t \geq 0)$ is a (true) martingale adapted to $(\mathcal{F}_t : t \geq 0)$. As in Section 3, we can define the probability $\tilde{\mathrm{P}}_x$ via

$$\tilde{\mathrm{P}}_x(A) = \mathrm{E}_x I(A) M(t)$$

for $A \in \mathcal{F}_t$. Applying the optional sampling theorem, we find that

$$\begin{aligned}
\tilde{\mathrm{P}}_x(\tau_b > t) &= \mathrm{E}_x I(\tau_b > t) M(t) \\
&= \mathrm{E}_x I(\tau_b > t) M(t) + \mathrm{E}_x I(\tau_b \leq t) M(\tau_b) \\
&= \mathrm{E}_x M(t \wedge \tau_b) = 1,
\end{aligned}$$

for all $t > 0$. Here the second equality follows from the fact that $v(X(\tau_b)) = v(b) = 0$ and the last equality holds by the optional sampling theorem. Hence, $\tau_b = \infty$ $\tilde{\mathrm{P}}_x$-a.s. (as expected). So,

$$\mathrm{P}_x(\tau_b > t) = \tilde{\mathrm{E}}_x M(t)^{-1} = e^{-\lambda t} \tilde{\mathrm{E}}_x \frac{v(X(0))}{v(X(t))}, \tag{4.2}$$

where $\tilde{\mathrm{E}}_x(\cdot)$ is the expectation operator associated with $\tilde{\mathrm{P}}_x$. Assume, temporarily, that X has a stationary distribution $\tilde{\pi}$ under $\tilde{\mathrm{P}}_x$ for which

$$\tilde{\mathrm{E}}_x \frac{v(X(0))}{v(X(t))} \to v(x) \int_{[0,b)} v^{-1}(y)\,\tilde{\pi}(\mathrm{d}y) \tag{4.3}$$

as $t \to \infty$. In view of (4.2) and (4.3), we can therefore characterize λ via

$$\lambda = \sup\{\theta : \mathrm{E}_x e^{\theta\tau_b} < \infty\}. \tag{4.4}$$

So, the correct choice of λ (chosen from the spectrum of the eigenvalue problem (4.1)) should be computable from $u^*(x) = \mathrm{E}_x \exp(\theta\tau_b)$. The function $u^* = (u^*(x) : 0 \le x \le b)$ must clearly be positive and decreasing on $[0, b]$ for positive θ. Note that for any solution u to

$$\mathcal{L}u = -\theta u \quad \text{on } [0, b], \tag{4.5}$$

subject to $u(b) = 1$ and $u'(0) = 0$, the process

$$e^{\theta(t\wedge\tau_b)} u(X(t \wedge \tau_b))$$

is a martingale adapted to $(\mathscr{F}_t : t \ge 0)$. If, in addition, u is positive, then

$$\mathrm{E}_x e^{\theta\tau_b} I(\tau_b \le t) \le u(x),$$

so that the Monotone Convergence Theorem implies that $\infty > u(x) \ge \mathrm{E}_x e^{\theta\tau_b}$. So, $\lambda \ge \lambda_0 = \sup\{\theta > 0 : (4.5)$ has a positive decreasing solution u satisfying $u'(0) = 0$ and $u(b) = 1\}$. Let

$$\begin{aligned}
\mathscr{D}_1 &= \{(\mu, b) : \mu \ge 0\} \\
\mathscr{D}_2 &= \{(\mu, b) : \mu < 0, b\mu + \sigma^2 > 0\} \\
\mathscr{D}_3 &= \{(\mu, b) : \mu < 0, b\mu + \sigma^2 < 0\} \\
\mathscr{D}_4 &= \{(\mu, b) : \mu < 0, b\mu + \sigma^2 = 0\}
\end{aligned}$$

Proposition 2 *Suppose $\theta > 0$. The differential equation (4.5), subject to $u'(0) = 0$ and $u(b) = 1$, has a positive decreasing solution if and only if $\theta < \lambda_0$, where*

a.) For $(\mu, b) \in \mathscr{D}_i (i = 1, 2)$, $\lambda_0 = \frac{\mu^2+\xi_*^2}{2\sigma^2}$, where ξ_* is the unique root in $(0, \frac{\pi\sigma^2}{b})$ of the equation

$$\frac{b\xi}{\sigma^2} + \arccos\left(\frac{\mu}{\sqrt{\mu^2 + \xi^2}}\right) = \pi.$$

b.) For $(\mu, b) \in \mathscr{D}_3$, $\lambda_0 = \frac{\mu^2-\beta_*^2}{2\sigma^2}$, where β_* is the unique root in $(0, -\mu)$ of the equation

$$\frac{1}{\beta} \log\left(\frac{-\mu + \beta}{-\mu - \beta}\right) = \frac{2b}{\sigma^2}.$$

c.) For $(\mu, b) \in \mathscr{D}_4$, $\lambda_0 = \frac{\mu^2}{2\sigma^2}$.

The above discussion suggests that we can identify the eigenvalue λ for (4.1) corresponding to a positive eigenfunction v as $\lambda = \lambda_0$. Theorem 4 proves that there does indeed exist a positive eigenfunction for (4.1) corresponding to λ_0, so that $\lambda = \lambda_0$ (rigorously).

Theorem 4 *a.) For $(\mu, b) \in \mathscr{D}_i (i = 1, 2)$, the solution v to (4.1) with $\lambda = \lambda_0$ is*

$$v(x) = e^{-\frac{\mu}{\sigma^2}x}\left[\cos\left(\frac{\xi_*}{\sigma^2}x\right) + \frac{\mu}{\xi_*}\sin\left(\frac{\xi_*}{\sigma^2}x\right)\right].$$

b.) For $(\mu, b) \in \mathscr{D}_3$, the solution v to (4.1) with $\lambda = \lambda_0$ is

$$v(x) = \frac{1}{2\beta_*}e^{-\frac{\mu}{\sigma^2}x}\left[(\beta_* - \mu)e^{-\frac{\beta_*}{\sigma^2}x} + (\beta_* + \mu)e^{\frac{\beta_*}{\sigma^2}x}\right].$$

c.) For $(\mu, b) \in \mathscr{D}_4$, the solution v to (4.1) with $\lambda = \lambda_0$ is

$$v(x) = e^{-\frac{\mu}{\sigma^2}x}\left(\frac{\mu}{\sigma^2}x + 1\right).$$

In each case, the solution v is positive on $[0, b]$.

Following the same argument as in Section 3, it can be shown that if $\tilde{X}$ has the law of X under $\tilde{\mathrm{P}}_x$, $\tilde{X}$ satisfies the SDE

$$\begin{aligned} \mathrm{d}\tilde{X}(t) &= \left(\mu + \frac{v'(\tilde{X}(t))}{v(\tilde{X}(t))}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}B(t) + \mathrm{d}\tilde{L}(t), \\ &\triangleq \tilde{\mu}(\tilde{X}(t))\mathrm{d}t + \sigma\mathrm{d}B(t) + \mathrm{d}\tilde{L}(t), \end{aligned} \tag{4.6}$$

subject to $\tilde{X}(0) = x$, where $\tilde{L}(\cdot)$ is the local time process at the origin associated with $\tilde{X}$. A related calculation in which the spectral representation of the transition density for reflected Brownian motion with one reflecting and one absorbing barrier is derived, using a purely analytical separation-of-variables argument, can be found in [16].

We turn next to the equilibrium behavior of $\tilde{X}$. Let $\tilde{\mathcal{L}}$ be the second order differential operator given by

$$\tilde{\mathcal{L}} = \tilde{\mu}(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

For any function f that is twice differentiable on $[0, b]$, Itô's formula establishes that

$$f(\tilde{X}(t)) - \int_0^t (\tilde{\mathcal{L}}f)(\tilde{X}(s))\,\mathrm{d}s - f'(0)\tilde{L}(t)$$

is a (true) martingale. It follows that if $\tilde{\pi}$ is a stationary distribution of $\tilde{X}$, then

$$\int_{[0,b)} \tilde{\pi}(\mathrm{d}x)(\tilde{\mathcal{L}}f)(x) = 0 \tag{4.7}$$

for all such functions f satisfying $f'(0) = 0$; conversely, if a probability $\tilde{\pi}$ satisfies (4.7), $\tilde{\pi}$ is stationary for $\tilde{X}$ (see [7]). If $\tilde{\pi}$ has a twice continuously differentiable density $\tilde{p}$, integration-by-parts guarantees that

$$\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{p}(x) - \frac{\mathrm{d}}{\mathrm{d}x}(\tilde{\mu}(x)p(x)) = 0, \tag{4.8}$$

subject to the boundary condition

$$\tilde{\mu}(0)\tilde{p}(0) - \frac{\sigma^2}{2}\tilde{p}'(0) = 0. \tag{4.9}$$

The equations (4.8) and (4.9) can be solved explicitly and the result can be found in Theorem 5 below; we omit the details. One can then verify directly that $\tilde{\pi}(\mathrm{d}x) = \tilde{p}(x)\mathrm{d}x$ satisfies (4.7), from which it follows that $\mathrm{E}f(\tilde{X}(t)) = \mathrm{E}f(\tilde{X}(0))$, provided that f is twice differentiable on $[0,b]$ with $f'(0) = 0$ and $\tilde{X}(0)$ has distribution $\tilde{\pi}$. Since one can uniformly approximate indicator functions of the form $I(x \geq c)$ for $0 < c \leq b$ by such functions f, this establishes that $\tilde{\pi}$ is indeed a stationary distribution for $\tilde{X}$.

Theorem 5 *The process $\tilde{X}$ has a stationary distribution given by*

$$\tilde{\pi}(\mathrm{d}x) = \frac{e^{\frac{2\mu}{\sigma^2}x}v^2(x)\mathrm{d}x}{\displaystyle\int_{[0,b)} e^{\frac{2\mu}{\sigma^2}y}v^2(y)\,\mathrm{d}x}$$

for $0 \leq x < b$.

The distribution $\tilde{\pi}$ is the so-called quasi-stationary distribution associated with conditioning X on not hitting b. Since $\tilde{X}$ is positive recurrent and admits coupling (in particular, a $\tilde{\mathrm{P}}_x$-version couples with a $\tilde{\mathrm{P}}_{\tilde{\pi}}$-version when the "upper process" hits the origin), (4.3) follows. Hence,

$$\mathrm{P}_x(\tau_b > t) \sim e^{-\lambda t}v(x) \int_{[0,b)} v^{-1}(y)\,\tilde{\pi}(\mathrm{d}y)$$

as $t \to \infty$, proving that λ can be characterized via (4.4).

5 Proofs

Lemma 1 *a.) If $(\theta,\mu,b) \in \mathscr{R}_i$ $(i = 1,3)$, (3.6) has a unique root in $\mathscr{I}_i$.*

b.) If $(\theta,\mu,b) \in \mathscr{R}_i$ $(i = 2,4)$, (3.7) has a unique root in $(0, \frac{\pi\sigma^2}{b})$.

For part a.), consider first the region $\mathscr{R}_1$. Let

$$f(z) = \frac{1}{z}\log\left(\frac{(z-\mu)(z+\mu+\theta\sigma^2)}{(z+\mu)(z-\mu-\theta\sigma^2)}\right).$$

Then, $f'(z) = h(z)/z^2$, where

$$h'(z) = \frac{4z^2}{(z^2-\mu^2)^2(z^2-(\mu+z\sigma^2)^2)^2}k(z)$$

and $k(z) = \theta\sigma^2[(z^2+\mu(\mu+\theta\sigma^2))^2 - \mu(\mu+\theta\sigma^2)(2\mu+\theta\sigma^2)^2]$. If $\mu \geq 0$, k is increasing on $\mathscr{I}_1$ so $k(z) \geq k(\mu+\theta\sigma^2) > 0$ there. Because $h(\infty) = 0$, it follows that $h(z) < 0$ on $\mathscr{I}_1$, so that f is decreasing on that interval. But

$$\lim_{z\downarrow\mu+\theta\sigma^2} f(z) = \infty \quad \text{and} \quad \lim_{z\uparrow\infty} f(z) = 0.$$

Hence, f has a unique root $\beta \in \mathscr{I}_1$ satisfying $f(\beta) = \frac{2b}{\sigma^2}$.

Similar arguments establish that if $\mu < 0$ and $\theta \in (0, -\frac{\mu}{\sigma^2})$, then $f(\beta) = \frac{2b}{\sigma^2}$ has a unique root $\beta \in (-\mu, \infty)$, whereas if $\mu < 0$ and $\theta \geq -\frac{\mu}{\sigma^2}$, then (3.6) has a unique root in $\mathscr{I}_1$.

We turn next to the region $\mathscr{R}_3$. By a similar argument as that for for $\mathscr{R}_1$, it can be seen that f is increasing on $(0, -\mu)$ when $\theta < 0$ and $\mu < 0$. Since $f(\infty) = 0$ and

$$\lim_{z \downarrow 0} f(z) = -\frac{2\theta\sigma^2}{\mu(\mu + \theta\sigma^2)},$$

evidently (3.6) has a unique root in $(0, -\mu)$ when $\lim_{z\downarrow 0} f(z) < \frac{2b}{\sigma^2}$ (i.e. $b\mu(\mu + \theta\sigma^2) > -\theta\sigma^4$). On the other hand, if $\mu > 0$ and $\theta \in (-\frac{\mu}{\sigma^2}, 0)$, (3.6) has a unique root on $(0, \mu + \theta\sigma^2)$ whenever $b\mu(\mu + \theta\sigma^2) > -\theta\sigma^4$.

For part b.), we let

$$g(z) = \frac{bz}{\sigma^2} - \arccos\left(\frac{z^2 + \mu^2 + \mu\theta\sigma^2}{\sqrt{(z^2 + \mu^2 + \mu\theta\sigma^2)^2 + z^2\theta^2\sigma^4}}\right).$$

Then, $g'(z) = \frac{l(z)}{\sigma^2[(z^2+\mu^2+\mu\theta\sigma^2)^2+z^2\theta^2\sigma^4]}$, where

$$l(z) = z^4 b + z^2[2b\mu(\mu + \theta\sigma^2) + b\theta^2\sigma^4 - \theta\sigma^4] + \mu(\mu + \theta\sigma^2)[b\mu(\mu + \theta\sigma^2) + \theta\sigma^4].$$

Consider first the region $\mathscr{R}_2$ in which $\mu(\mu + \theta\sigma^2) \leq 0$. Then $l(z) > 0$ for $z > 0$, so $g(z)$ is increasing on $(0, \infty)$. But $\lim_{z\to\infty} g(z) = \infty$ and

$$\lim_{z\downarrow 0} g(z) = \begin{cases} -\pi, & \text{if } \mu(\mu + \theta\sigma^2) < 0 \\ -\frac{\pi}{2}, & \text{if } \mu(\mu + \theta\sigma^2) = 0 \end{cases}$$

Hence, g has a unique root $\xi > 0$ satisfying $g(\xi) = 0$.

We turn next to the region $\mathscr{R}_4$. Then clearly $l(z)$ is increasing on $(0, \infty)$ (for the coefficients of z^4 and z^2 are both positive). Note that

$$\lim_{z\downarrow 0} l(z) = \mu(\mu + \theta\sigma^2)[b\mu(\mu + \theta\sigma^2) + \theta\sigma^4] < 0.$$

So there exists a unique $\tilde{\xi} > 0$ such that $l(\tilde{\xi}) = 0$ and that $g(z)$ is decreasing on $(0, \tilde{\xi})$ and is increasing on $(\tilde{\xi}, \infty)$. But

$$\lim_{z\downarrow 0} g(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} g(z) = \infty.$$

Hence, (3.7) has a unique positive root.

Finally, note that the arccos$(\cdot)$ term of g is in $(0, \pi)$, from which it follows that $\frac{b\xi}{\sigma^2} < \pi$ and thus $\xi \in (0, \frac{\pi\sigma^2}{b})$. □

Proof of Theorem 1

The equation (3.5) is a linear differential equation with constant coefficients, so we seek a solution of the form $v(x) = Ae^{\gamma_1 x} + Be^{\gamma_2 x}$. The quantities γ_1, γ_2 arise as roots of the quadratic equation

$$\gamma^2 + \frac{2\mu}{\sigma^2}\gamma - \frac{2\psi}{\sigma^2} = 0. \tag{5.1}$$

The discriminant of the quadratic is $\Delta = \frac{4(\mu^2+2\psi\sigma^2)}{\sigma^4}$. The form of the solution to (3.5) depends critically on the sign of Δ.

Case 1: $\Delta > 0$.

In this case, (5.1) has two distinct real roots $\gamma_1 = -\frac{\mu+\beta}{\sigma^2}$ and $\gamma_2 = -\frac{\mu-\beta}{\sigma^2}$, where $\beta = \sqrt{\mu^2 + 2\psi\sigma^2} > 0$. The boundary conditions $v'(0) = 0$ and $v(0) = 1$ identify A and B as $A = \gamma_2(\gamma_2 - \gamma_1)^{-1}$ and $B = -\gamma_1(\gamma_2 - \gamma_1)^{-1}$. The third boundary condition $v'(b) = \theta v(b)$ leads to

$$e^{(\gamma_2-\gamma_1)b} = \frac{\gamma_2(\gamma_1 - \theta)}{\gamma_1(\gamma_2 - \theta)}$$

or, equivalently, (3.6). Lemma 1 establishes that when $(\theta, \mu, b) \in \mathscr{R}_1$, (3.6) has a unique root β on $\mathscr{I}_1$ for which $\psi = \frac{\beta^2-\mu^2}{2\sigma^2}$. It follows that $(\gamma_2 - \gamma_1)v'(x) = \gamma_1\gamma_2(e^{\gamma_1 x} - e^{\gamma_2 x}) = \frac{2\psi}{\sigma^2}(e^{\gamma_1 x} - e^{\gamma_2 x}) < 0$ with $v(b) = e^{\gamma_1 b}\frac{\beta-\mu}{\beta-\mu-\theta} > 0$, and hence v is positive on $[0, b]$.

When $(\theta, \mu, b) \in \mathscr{R}_3$, Lemma 1 proves that (3.6) has a unique root β on $\mathscr{I}_3$. In this region, $\psi \leq 0$ so $v'(x) \geq 0$ for $x \in [0, b]$. Since $v(0) = 1$, v is therefore positive on $[0, b]$.

Case 2: $\Delta < 0$.

In this case, (5.1) has two distinct complex roots and v can be written in the form

$$v(x) = e^{-\frac{\mu}{\sigma^2}x}\left[A\cos\left(\frac{\xi x}{\sigma^2}\right) + B\sin\left(\frac{\xi x}{\sigma^2}\right)\right], \tag{5.2}$$

where $\xi = \sqrt{-(\mu^2 + 2\psi\sigma^2)} > 0$. The boundary conditions $v(0) = 1$ and $v'(0) = 0$ ensure that $A = 1$ and $B = \frac{\mu}{\xi}$. The third boundary condition $v'(b) = \theta v(b)$ yields the equality

$$0 = (\xi^2 + \mu^2 + \mu\theta\sigma^2)\sin\left(\frac{\xi b}{\sigma^2}\right) + \xi\theta\sigma^2\cos\left(\frac{\xi b}{\sigma^2}\right),$$

from which it follows that (3.7) holds. Lemma 1 establishes that there exists a unique root lying on $(0, \frac{\pi\sigma^2}{b})$ to (3.7) when (θ, μ, b) lies in either $\mathscr{R}_2$ or $\mathscr{R}_4$. Recalling that $\sin(w + v) = \cos(w)\sin(v) + \sin(w)\cos(v)$ for $w, v \in \mathbb{R}$, we can rewrite (5.2) as

$$v(x) = e^{-\frac{\mu}{\sigma^2}x}\sqrt{1 + \left(\frac{\mu}{\xi}\right)^2}\sin\left(\frac{\xi x}{\sigma^2} + \alpha\right),$$

where $\alpha = \arccos(\mu(\xi^2 + \mu^2)^{-\frac{1}{2}}) \in (0, \pi)$. The function v is therefore positive provided that $\frac{\xi b}{\sigma^2} + \alpha < \pi$. Given that $0 < \frac{\xi b}{\sigma^2}, \alpha < \pi$, it is sufficient to prove the inequality $\sin\left(\frac{\xi b}{\sigma^2} + \alpha\right) > 0$. But this is clear, given that

$$\begin{aligned}\sin\left(\frac{\xi b}{\sigma^2} + \alpha\right) &= \cos\left(\frac{\xi b}{\sigma^2}\right)\sin(\alpha) + \sin\left(\frac{\xi b}{\sigma^2}\right)\cos(\alpha)\\ &= \frac{\xi^2 + \mu^2 + \mu\theta\sigma^2}{\sqrt{(\xi^2 + \mu^2 + \mu\theta\sigma^2)^2 + \xi^2\theta^2\sigma^4}} \cdot \frac{\xi}{\sqrt{\xi^2 + \mu^2}}\\ &\quad - \frac{\xi\sigma^2}{\sqrt{(\xi^2 + \mu^2 + \mu\theta\sigma^2)^2 + \xi^2\theta^2\sigma^4}} \cdot \frac{\mu}{\sqrt{\xi^2 + \mu^2}}.\end{aligned}$$

Case 3: $\Delta = 0$.

In this case, $\psi = -\frac{\mu^2}{2\sigma^2}$, and v takes the form $v(x) = e^{-\frac{\mu}{\sigma^2}x}(A + Bx)$. In view of the fact that $v'(0) = 0$ and $v(0) = 1$, $A = 1$ and $B = \frac{\mu}{\sigma^2}$. Because $v'(b) = \theta v(b)$, we conclude that

$$v(x) = e^{-\frac{\mu}{\sigma^2}x}\left(1 + \frac{\mu x}{\sigma^2}\right),$$

which is clearly positive on $[0, b]$. □

Proof of Theorem 2

Given the Gärtner-Ellis theorem and the smoothness of ψ, it remains only to prove that there exists a unique root θ_γ for each $\gamma > 0$. It is straightforward to verify that $\beta(\theta) \sim \sigma^2\theta$ as $\theta \to \infty$, so that $\theta^{-1}\psi(\theta) = \theta^{-1}(\psi(\theta) - \psi(0)) \to \infty$ as $\theta \to \infty$. The mean value theorem implies the existence of $\tilde{\theta} \in [0, \theta]$ such that $\psi'(\tilde{\theta}) = \theta^{-1}\psi(\theta)$ and hence $\overline{\lim}_{\theta\to\infty}\psi'(\theta) = \infty$. On the other hand, it is easily seen that $\psi(\theta)$ is bounded below as $\theta \to -\infty$, so that $\underline{\lim}_{\theta\to\infty}\psi'(\theta) = 0$. Since ψ' is strictly increasing and continuous, this guarantees existence of a unique solution to $\psi'(\theta_\gamma) = \gamma$ for each $\gamma > 0$. □

Lemma 2 *Suppose $\mu < 0$. Let $f(z) = z^{-1}\log\left(\frac{-\mu+z}{-\mu-z}\right)$.*

a.) f is increasing on $(0, -\mu)$.

b.) For $(\mu, b) \in \mathscr{D}_i$ $(i = 2, 4)$, $f(z) > \frac{2b}{\sigma^2}$ for $z > 0$.

c.) For $(\mu, b) \in \mathscr{D}_3$, there exists a unique root β_* in $(0, -\mu)$ satisfying $f(\beta_*) = \frac{2b}{\sigma^2}$.

Note that $f'(z) = z^{-2}h(z)$, where $h(z) = -\left(\frac{2\mu z}{\mu^2-z^2} + \log\left(\frac{-\mu+z}{-\mu-z}\right)\right)$. So $h(0) = 0$ and $h'(z) = -\frac{2\mu z^2}{\mu^2-z^2} > 0$. Hence, f is increasing on $(0, -\mu)$. But

$$\lim_{z\downarrow 0} f(z) = -\frac{2}{\mu} \quad \text{and} \quad \lim_{z\uparrow -\mu} f(z) = \infty.$$

Hence, for $(\mu, b) \in \mathscr{D}_i$ $(i = 2, 4)$, $f(z) > -\frac{2}{\mu} \geq \frac{2b}{\sigma^2}$ whereas for $(\mu, b) \in \mathscr{D}_3$, there exists a unique positive β_* root such that $f(\beta_*) = \frac{2b}{\sigma^2}$. □

Lemma 3 *Let $g(z) = \frac{bz}{\sigma^2} + \arccos\left(\frac{\mu}{\sqrt{\mu^2+z^2}}\right)$*

a.) For $(\mu, b) \in \mathscr{D}_i$ $(i = 1, 2)$, there exists a unique root ξ_* in $(0, \frac{\pi\sigma^2}{b})$ satisfying $g(\xi_*) = \pi$. Moreover, $g(z) < \pi$ for $z \in (0, \xi_*)$ and $g(z) > \pi$ for $z \in (\xi_*, \infty)$.

b.) For $(\mu, b) \in \mathscr{D}_i$ $(i = 3, 4)$, $g(z) > \pi$ for $z > 0$.

For part a.), consider first the region $\mathscr{D}_1$. Note that $g'(z) = \frac{b}{\sigma^2} + \frac{\mu}{\mu^2+z^2}$. If $\mu \geq 0$, $g'(z) > 0$ for $z > 0$. But

$$\lim_{z\downarrow 0} g(z) = \begin{cases} 0, & \text{if } \mu > 0 \\ \frac{\pi}{2}, & \text{if } \mu = 0 \end{cases} \qquad \text{and} \qquad \lim_{z\to\infty} g(z) = \infty.$$

Hence, g has a unique positive root ξ_* such that $g(\xi_*) = \pi$.

If $\mu < 0$, $\lim_{z\downarrow 0} g(z) = \pi$ and clearly $g'(z)$ is increasing on $(0, \infty)$. Note that

$$\lim_{z\downarrow 0} g'(z) = b\sigma^{-2} + \mu^{-1} \quad \text{and} \quad \lim_{z\to\infty} g'(z) = b\sigma^{-2} > 0.$$

Hence, for $(\mu, b) \in \mathscr{D}_2$, there exists a unique $\hat{\xi} > 0$ such that $g'(\hat{\xi}) = 0$ for which $g(z)$ is decreasing on $(0, \hat{\xi})$ and increasing on $(\hat{\xi}, \infty)$. So there exists a unique $\xi_* > 0$ such that $g(\xi_*) = \pi$ for which $g(z) < \pi$ for $z \in (0, \xi_*)$ and $g(z) > \pi$ for $z \in (\xi_*, \infty)$. Since the $\arccos(\cdot)$ term of g is in $(0, \pi)$, we conclude that $\xi_* \in (0, \frac{\pi\sigma^2}{b})$.

For part b.), if $(\mu, b) \in \mathscr{D}_3 \cup \mathscr{D}_4$, $g'(z) > g'(0+) = b\sigma^{-2} + \mu^{-1} \geq 0$ for $z > 0$ and thus $g(z) > \pi$ for $z > 0$. $\square$

Proof of Proposition 2

Since a solution u satisfies the boundary condition $u(b) = 1$, the positivity of u is implied by the decreasing monotonicity. So we need to prove that there exists a decreasing solution to (4.5) if and only if $\theta \in (0, \lambda_0)$.

As in Theorem 1, the solution to the linear differential equation depends critically on the sign of $\Delta = \frac{4(\mu^2 - 2\theta\sigma^2)}{\sigma^2}$, the discriminant of the quadratic equation

$$\gamma^2 + \frac{2\mu}{\sigma^2}\gamma + \frac{2\theta}{\sigma^2} = 0. \tag{5.3}$$

If $\Delta \neq 0$, (5.3) has two distinct (possibly complex) roots γ_1 and γ_2 and $u(x) = Ae^{\gamma_1 x} + Be^{\gamma_2 x}$.

For part a.), note that if $\theta \in (0, \frac{\mu^2}{2\sigma^2})$, $\gamma_1 = -\frac{\mu+\beta}{\sigma^2}$ and $\gamma_2 = -\frac{\mu-\beta}{\sigma^2}$, where $\beta = \sqrt{\mu^2 - 2\theta\sigma^2} > 0$. Given the boundary conditions,

$$u'(x) = \frac{\gamma_1\gamma_2(e^{\gamma_1 x} - e^{\gamma_2 x})}{\gamma_2 e^{\gamma_1 b} - \gamma_1 e^{\gamma_2 b}}. \tag{5.4}$$

Clearly, $u'(x) < 0$ on $[0, b]$ if and only if the denominator is positive. But this is trivial for $(\mu, b) \in \mathscr{D}_1$ (since $\gamma_1 < \gamma_2 < 0$). For $(\mu, b) \in \mathscr{D}_2$, note that $\beta \in (0, -\mu)$, from which it follows from Lemma 2 that $\beta^{-1}\log(\frac{-\mu+\beta}{-\mu-\beta}) > \frac{2b}{\sigma^2}$ (which is equivalent to the positivity of the denominator since $0 < \gamma_1 < \gamma_2$ for $(\mu, b) \in \mathscr{D}_2$). If $\theta = \frac{\mu^2}{2\sigma^2}$,

$$u'(x) = \frac{e^{\frac{\mu}{\sigma^2}(b-x)}}{\mu b + \sigma^2}\left(-\frac{\mu^2 x}{\sigma^2}\right), \tag{5.5}$$

so that $u'(x) < 0$ on $[0, b]$ in view of the positivity of $\mu b + \sigma^2$ for $(\mu, b) \in \mathscr{D}_1 \cup \mathscr{D}_2$. Finally, for $\theta \in (\frac{\mu^2}{2\sigma^2}, \lambda_0)$,

$$u'(x) = \frac{e^{\frac{\mu}{\sigma^2}(b-x)}}{\xi\cos(\frac{\xi b}{\sigma^2}) + \mu\sin(\frac{\xi b}{\sigma^2})}\left(-\frac{\xi^2 + \mu^2}{\sigma^2}\right)\sin\left(\frac{\xi x}{\sigma^2}\right), \tag{5.6}$$

where $\xi = \sqrt{2\theta\sigma^2 - \mu^2} \in (0, \xi_*)$. Lemma 3 establishes that $0 < \frac{\xi b}{\sigma^2} + \varphi < \pi$, where $\varphi = \arccos(\frac{\mu}{\sqrt{\xi^2+\mu^2}})$. Hence, $\xi\cos(\frac{\xi b}{\sigma^2}) + \mu\sin(\frac{\xi b}{\sigma^2}) = \sqrt{\xi^2 + \mu^2}\sin(\frac{\xi b}{\sigma^2} + \varphi) > 0$, so that $u'(x) < 0$ for $x \in [0, b]$.

It remains to show that u is not decreasing when $\theta \geq \lambda_0$. Here, $\Delta < 0$ and u' is given by (5.6), where $\xi \in [\xi_*, \infty)$. If $\xi > \frac{\pi\sigma^2}{b}$, then u' has mixed sign on $[0, b]$, whereas if $\xi \in [\xi_*, \frac{\pi\sigma^2}{b}]$, then

$\frac{\xi b}{\sigma^2}+\varphi<2\pi$, so that Lemma 3 implies that $\xi\cos(\frac{\xi b}{\sigma^2})+\mu\sin(\frac{\xi b}{\sigma^2})\le 0$, so that u is non-decreasing on $[0,b]$.

For part b.), if $\theta\in(0,\lambda_0)$, u' is given by (5.4), where $\beta\in(\beta_*,-\mu)$. It follows from Lemma 2 that $\beta^{-1}\log(\frac{-\mu+\beta}{-\mu-\beta})>\frac{2b}{\sigma^2}$ for $(\mu,b)\in\mathscr{D}_3$, so that $u'(x)<0$ for $x\in[0,b]$.

For the converse, we now show that u is not decreasing when $\theta\ge\lambda_0$. If $\theta\in[\lambda_0,\frac{\mu^2}{2\sigma^2})$, u' is given by (5.4), where $\beta\in(0,\beta_*]$. Lemma 2 implies that $\beta^{-1}\log(\frac{-\mu+\beta}{-\mu-\beta})$
$\le\frac{2b}{\sigma^2}$ for $(\mu,b)\in\mathscr{D}_3$. Hence, $\gamma_2e^{\gamma_1b}\le\gamma_1e^{\gamma_2b}$, so that u is non-decreasing on $[0,b]$. If $\theta=\frac{\mu^2}{2\sigma^2}$, u' is given by (5.5). But clearly $\mu b+\sigma^2<0$, so that u is increasing on $[0,b]$. If $\theta>\frac{\mu^2}{2\sigma^2}$, u is given by (5.6). Lemma 3 implies that $\frac{\xi b}{\sigma^2}+\varphi>\pi$ for $(\mu,b)\in\mathscr{D}_3$. With a similar argument as for part a.), we conclude that u is not decreasing.

For part c.), if $\theta<\frac{\mu^2}{2\sigma^2}$, u' is given by (5.4),where $\beta\in(0,-\mu)$. Lemma 2 implies that $\beta^{-1}\log(\frac{-\mu+\beta}{-\mu-\beta})>\frac{2b}{\sigma^2}$, for $(\mu,b)\in\mathscr{D}_4$. Hence, $u'(x)<0$ for $x\in[0,b]$.

It remains to show that u is not decreasing when $\theta\ge\lambda_0$. If $\theta=\frac{\mu^2}{2\sigma^2}$, u' is given by (5.5). For $(\mu,b)\in\mathscr{D}_4$, $b\mu+\sigma^2=0$. So u is not decreasing. If $\theta>\frac{\mu^2}{2\sigma^2}$, u' is given by (5.6). Lemma 3 implies that $\frac{\xi b}{\sigma^2}+\varphi>\pi$ for $(\mu,b)\in\mathscr{D}_4$. With a similar argument as for part a.), we conclude that u is not decreasing. □

Proof of Theorem 4

With the definitions of λ_0, β_* and ξ_*, one can easily verify the given formula $v(x)$ does satisfy (4.1) with the boundary condition $v(0)=1$, $v'(0)=0$, and $v(b)=0$. So we only check the positivity of $v(x)$ on $[0,b)$ in the following proof.

For part a.), since $\xi_*\in(0,\frac{\pi\sigma^2}{b})$,

$$v'(x)=e^{-\frac{\mu}{\sigma^2}x}\left(-\frac{\xi_*^2+\mu^2}{\sigma^2\xi_*}\right)\sin\left(\frac{\xi_*}{\sigma^2}x\right)<0.$$

For part b.), since $\beta_*\in(0,-\mu)$,

$$v'(x)=e^{-\frac{\mu}{\sigma^2}x}\left(\frac{\mu^2-\beta_*^2}{\sigma^2}\right)\left(e^{-\frac{\beta_*}{\sigma^2}x}-e^{\frac{\beta_*}{\sigma^2}x}\right)<0.$$

For part c.),

$$v'(x)=e^{-\frac{\mu}{\sigma^2}x}\left(-\frac{\mu^2}{\sigma^4}x\right)<0.$$

Therefore, $v(x)>v(b)=0$ for $x\in[0,b)$. □

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