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Approximations for the distribution of perpetuities with small discount rates

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Abstract

Perpetuities (i.e., random variables of the form $D = \int_0^\infty e^{-\Gamma(t-)} d\Lambda(t)$ play an important role in many application settings. We develop approximations for the distribution of D when the "accumulated short rate process", Γ, is small. We provide: (1) characterizations for the distribution of D when Γ and Λ are driven by Markov processes; (2) general sufficient conditions under which weak convergence results can be derived for D, and (3) Edgeworth expansions for the distribution of D in the iid case and the case in which Λ is a Levy process and the interest rate is a function of an ergodic Markov process.

KEYWORDS

central limit theorem, Edgeworth expansions, stochastic discounted reward

1 | INTRODUCTION

It is our pleasure to contribute to this special issue in honor of A. F. Veinott. Prof. Veinott's interests spanned many areas of operations research. This article contributes to the study of infinite horizon stochastic discounted rewards with small discount rates. This topic is closely related to the study of concepts such as strong present value optimality, which was introduced in Blackwell (1962) and extensively studied and extended in Veinott Jr (1969). While the focus in their work was on expected values, we concentrate on the study of the distribution of the perpetuity in the small discount rate setting.

Given an $\mathbb{R}^2$-valued cadlag (right continuous paths with left limits) process $(\Lambda(t), \Gamma(t) : t \geq 0)$, the corresponding perpetuity is the random variable (rv)

$$D = \int_0^\infty \exp(-\Gamma(t_-))\, d\Lambda(t). \tag{1}$$

The simplest possible perpetuity (in continuous time) arises when Γ and Λ are differentiable with $d\Lambda(t)/dt = \dot{\Lambda}(t) \triangleq \lambda(t) = \lambda$ and $d\Gamma(t)/dt = \dot{\Gamma}(t) \triangleq \gamma(t) = \gamma$, in which case $D = \lambda/\gamma$. This arises, in the insurance setting, as the present value of a benefit payable to a policy holder in perpetuity at a fixed rate λ, under the assumption that the interest rate is fixed at the level γ. When considering an entire group of policy holders (as arises in managing a private pension fund), a more realistic formulation models the aggregate rate $\lambda(t)$ paid out by the company and the instantaneous "force of interest" (i.e., the "short rate") at time t as stochastic processes. This stochastic formulation has been followed, for example, by Dufresne (1990) for valuing a pension fund.

In many applications contexts, the expectation ED is of principal interest. For example, many economics and operations research models compute, as a key mathematical quantity, the infinite horizon net present value of an economic consumption/investment strategy. Such a net present value can be characterized as the expectation of a perpetuity. However, there are a number of important applications domains in which the entire distribution of D is relevant:

(a) The distribution of D plays an important role in the above pension fund valuation approach as introduced by Dufresne (1990), where it serves as a critical ingredient in computing critical rates that ensure that the fund is managed in a balanced manner relative to its actuarial liabilities; see Bédard and Dufresne (2001) for additional details.

(b) The stochastic equation

$$d\widetilde{D}(t) = \alpha(t)\widetilde{D}(t)dt + \lambda(t)dt \tag{2}$$

arises in several different settings. For example, if $\widetilde{D}(t)$ is the reserve of an insurance company that receives premiums at a rate of p dollars per unit time and pays out claims at a rate $c(t)$ at time t, then $\widetilde{D}(t)$ satisfies the above equation, with $\lambda(t) = p - c(t)$ and $\alpha(t)$ representing the rate of return on the invested risk reserve at time t. Harrison (1977) showed that when α is constant and deterministic, then the infinite horizon ruin probability $P\left(\inf_{t\geq 0}\widetilde{D}(t) < 0\right)$ for the insurance company can be computed in terms of the distribution of an appropriately defined perpetuity of the form (1). Paulsen (1998) extended this to the case of stochastic $\alpha(\cdot)$; see also Paulsen (1993), Nyrhinen (2001) and the work of Kluppelberg and Kostadinova (2007).

Note that if $\widetilde{D}(0) = 0$, the solution of (2) is given by

$$\widetilde{D}(t) = \int_0^t \exp\left(\int_u^t \alpha(s)ds\right)\lambda(u)du = \int_{-t}^0 \exp\left(\int_v^0 \alpha(r+t)dr\right)\lambda(v+t)dv$$

It follows that if $((\lambda(t), \alpha(t)) : t \in (-\infty, \infty))$ is strictly stationary with $\alpha(\cdot) = -\gamma(\cdot) < 0$ and $\lambda(\cdot)$ positive, then

$$\widetilde{D}(t) \overset{D}{=} \int_{-t}^0 \exp\left(\int_v^0 \alpha(r)ds\right)\lambda(v)dv \to \int_{-\infty}^0 \exp\left(-\int_v^0 \gamma(r)ds\right)\lambda(v)dv \triangleq \widetilde{D}(\infty)$$

as $t \nearrow \infty$ (where $\overset{D}{=}$ denotes equality in distribution). Hence, computing the equilibrium distribution of the solution to the stochastic Equation (2) is equivalent to calculating the distribution of a perpetuity.

An important special case is that of an ARCH (1) process $\left(\widetilde{D}_n : n \geq 0\right)$. Such a model satisfies a discrete-time analog of (2), namely

$$\widetilde{D}_{n+1} - \widetilde{D}_n = A_n\widetilde{D}_n + B_n, \tag{3}$$

where the sequence $((A_n, B_n) : n \geq 0)$ is independent and identically distributed (iid). This class of time series models is widely applied within the statistics and econometrics communities, and has been used to describe log-asset returns, exchange rates, inflation, and many other financial and economic time series; see Campbell et al. (1999), Shephard (1996) and Wilkie (1986). Again, under modest additional conditions, $\widetilde{D}_n \Rightarrow \widetilde{D}_\infty$ as $n \nearrow \infty$, where $\widetilde{D}_\infty$ is a perpetuity of the form (1).

(c) The distribution of the perpetuity D arises also in complexity theory (in the context of the so-called "Quicksort" algorithm) and analytic number theory; see Goldie and Grübel (1996) for details. Carmona et al. (2001) and Embrechts and Goldie (1994) describe several other applications, including mathematical physics and finance, where the distribution of D is relevant. Further connections and related notions are discussed in Diaconis and Freedman (1999).

This article is concerned with developing tools for computing and/or approximating the distribution of the perpetuity D. Our approximations are rigorously justified in the context of a small force of interest, which is a setting that arises often in the applications described before such as pension fund modeling, insurance, finance, and econometrics. Our main contributions include:

i) A derivation of the equations to be solved in the Markov setting when calculating both the Laplace transform of D (Theorem 1) and D's distribution function (Theorem 2).
ii) A central limit-type theorem (Theorem 4) for D that holds when the force of interest process $(\gamma(t) : t \geq 0)$ is small. In contrast to previous results in the literature, such as those by Nelson (1990), Bucklew et al. (1993), Forniari and Mele (1997) our assumptions do not require stationarity or that the accumulated force of interest and rewards satisfy a Brownian weak convergence limit, nor do we require strong assumptions on the discounting process as in Whitt (1972) or Gerber (1971) (where the force of interest is assumed to be constant).
iii) Edgeworth expansions for the distribution of D in the iid case (Theorem 5) and in the case in which Λ is a Levy process and $\gamma(t)$ is a function of a geometrically ergodic Markov process. The iid result covers, for example, the ARCH (1) process, while the Levy/Markov expansion (Theorem 6) is of particular interest when investment returns on the reserve are considered within the insurance context (since it is common to model pure claim processes as Levy motions and interest rates as functions of Markovian mean-reverting processes). Formal Edgeworth expansions are also given for more general Markov processes. Related results have been derived for autoregressive processes, in which the techniques are similar to the case of constant force of interest, see Miao et al. (2013).

It should further be noted that the tail asymptotics for a fixed (not necessarily small) interest rate have been developed for D in some special cases (such as independent increment processes in discrete settings or Levy-driven discount factors in the continuous case) by, for example, Goldie (1991), Kesten (1973) and Maulik and Zwart (2006). Large deviations results and sharp asymptotics are investigated in Blanchet (2004).

The rest of the article is organized as follows. Section 2 discusses the exact computation of the Laplace transform and distribution of D in the Markov setting. Section 3 develops sufficient conditions that guarantee weak convergence of D as the "interest rate" goes to zero. We formulate these results in an asymptotic environment in which we introduce a discount rate parameter α, consider an accumulated force of interest process of the form $(\alpha\Gamma(t) : t \geq 0)$, and send α to zero. The development of Edgeworth expansions for both the iid case and the case of Markov-driven discount and reward rates is given in our final section, namely Section 4.

2 | EXACT COMPUTATION

We develop here the equations that need to be solved in order to compute the Laplace transform of D, its moments, and the distribution of D in the Markov setting. In particular, we assume here that $\lambda(\cdot)$ and $\gamma(\cdot)$ take the form $\lambda(t) \triangleq \widetilde{\lambda}(Y(t))$ and $\gamma(t) \triangleq \widetilde{\gamma}(Y(t))$, where $Y = (Y(t) : t \geq 0)$ is a cadlag (i.e., right-continuous with left-limits) Ξ-valued Markov process with stationary transition probabilities and $\widetilde{\lambda}$ and $\widetilde{\gamma}$ are real-valued functions. We shall assume throughout this section that Ξ is a Polish space. For $y \in \Xi$, let $P_y(\cdot)$ and $E_y(\cdot)$ be the probability distribution and expectation operators on the path-space of Y, conditional on $Y(0) = y$.

We first provide an informal discussion of the equations that arise in computing the Laplace transform, moments, and distribution of D. Later in this section, we state and prove theorems that rigorously guarantee that solutions to these equations coincide with the expectations and probabilities we seek. Proceeding informally, let A_Y be the generator of Y and $D(A_Y)$ its corresponding domain; a precise definition will be given later in this section. For example, when Ξ is a finite state space and Y is a Markov jump process, A_Y is Y's associated rate matrix and $D(A_Y)$ is the set of all real-valued functions (encoded as column vectors) defined on Ξ.

The equation for the Laplace transform $\phi(y,\theta) \triangleq E_y \exp(\theta D)$ for θ non-positive:

Solve

$$(A\phi)(y,\theta) = \left(\partial_\theta \phi(y,\theta)\widetilde{\gamma}(y)\theta - \phi(y,\theta)\widetilde{\lambda}(y)\theta\right), \quad (4)$$

for $(y,\theta) \in \Xi \times \trianglelefteq -\infty, 0]$, subject to $\phi(\cdot)$ being positive and $\phi(y,0) = 1$ for $y \in \Xi$.

The equation for the k'th *moment* $\phi_k(y) \triangleq E_y D^k$:

Let $\phi_0 = 1$ and solve for $(\phi_m : 1 \leq m \leq k)$ recursively (in m) via the system of.

equations

$$(A\phi_m)(y) = m\phi_m(y)\widetilde{\gamma}(y) + \widetilde{\lambda}(y)\phi_{m-1}(y). \quad (5)$$

The equation for the distribution function $h(y,z) \triangleq P_y (D \leq z)$:

Solve

$$(Ah)(y,z) = (\widetilde{\lambda}(y) - z\widetilde{\gamma}(y))(\partial_z h)(y,z), \quad (y,z) \in \Xi \times \mathbb{R}, \quad (6)$$

subject to $0 \leq h(y,z) \leq 1$, such that $\lim_{z\to\infty} h(y,z) = 1$ and $\lim_{z\to-\infty} h(y,z) = 0$.

We now provide several theorems that offer sufficient conditions under which one is guaranteed that the solutions to (4)–(6) correspond to D's Laplace transform, its moments, and its distribution, respectively.

Given a Markov process $Y = (Y(t) : t \geq 0)$, we say that f belongs to the domain of the (extended) generator A_Y of the process Y and write $f \in D(A_Y)$ if there exists a function g for which the process

$$M(t) = f(Y(t)) - \int_0^t g(Y(u))du \quad (7)$$

is a local martingale with respect to P_y for each $y \in \Xi$. Furthermore, we then write $A_Y f \triangleq g$, where g is any given member selected from the class of functions that satisfy (7). For the Markov models that arise in the great majority of practical applications, it is straightforward to identify sufficient conditions ensuring that $f \in D(A_Y)$ and to compute $A_Y f$:

Markov jump processes: Suppose that Y is a non-explosive Markov jump process living on a discrete state space Ξ and possessing rate matrix $Q = (Q(x,y) : x,y \in \Xi)$. Then, any function f for which $\sum_y |Q(x,y)f(y)| < \infty$ for each $x \in \Xi$ lies in $D(A_Y)$ and $(A_Y f)(x) = \sum_y Q(x,y)f(y)$; see, for example, Ethier and Kurtz (1985), p. 327.

Stochastic differential equations (SDE's): Suppose that for each $y \in \mathbb{R}^d$, there exists a non-explosive solution Y to the SDE

$$dY(t) = a(Y(t))dt + b(Y(t))dB(t),$$

subject to the initial condition $Y(0) = y$. If f is twice differentiable on $\mathbb{R}^d$, then $f \in D(A_Y)$ and $A_Y f = Lf$, where L is the second-order differential operator given by

$$(Lf)(y) = a(y) \cdot \nabla f(y) + \frac{1}{2} Trace\left(b(y)b(y)^T D^2 f(y)\right),$$

where $D^2 f$ denotes the second derivative of f. See, for instance, Stroock and Varadhan (2006), chapters 5–7, or Rogers and Williams (1994), chapter 5, theorem 24.1.

A similar easily verified sufficient condition for determining when $f \in D(A_Y)$ and computing $A_Y f$ is available in the jump-diffusion context; see, for example, Jacod and Shiryaev (2003) pp. 155–159.

Define $Z(t) = (Y(t), \Gamma(t), D(t))$, where

$$D(t) = \int_0^t \exp(-\Gamma(s))\widetilde{\lambda}(Y(s))ds.$$

Under the assumptions we have made in this section, $Z = (Z(t) : t \geq 0)$ is a Markov process on $\Xi \times \mathbb{R} \times \mathbb{R}$. The first part of the following result provides a sufficient condition for a function to be in the domain $D(A_Z)$ and computes the action of A_Z on such functions in terms of the generator A_Y, while the second part shows that the local martingale property is inherited by a certain class of positive processes constructed from Z. Such conditions will be the key to proving the validity of Equations (4)–(6).

Lemma 1 *Let* $\kappa : \Xi \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be such that*:

i) *For each* $(r,\rho) \in \mathbb{R}\times\mathbb{R}$, *we have* $\kappa(\cdot,r,\rho) \in D(A_Y)$ *and* $\partial_r\kappa(\cdot,r,\rho)$ *and* $\partial_\rho\kappa(\cdot,r,\rho)$ *are continuous* on Ξ.

ii) *For every fixed* $y \in \Xi$, $\kappa(y,\cdot)$ *is continuously differentiable on* $\mathbb{R} \times \mathbb{R}$ *and* $(A_Y\kappa)(y,\cdot)$ *is continuous*.

If

$$\beta(Z(t)) = (A_Y\kappa)(Z(t)) + \partial_r\kappa(Z(t))\widetilde{\gamma}(Y(t)) + \partial_\rho\kappa(Z(t))e^{-\Gamma(t)}\widetilde{\lambda}(Y(t))$$

for $t \geq 0$, *then, for every* $z \in \Xi \times \mathbb{R} \times \mathbb{R}$, *the process* $(M_1(t) : t \geq 0)$ *defined* via

$$M_1(t) = \kappa(Z(t)) - \int_0^t \beta(Z(s))ds$$

is a local martingale, conditional on $Z(0) = z$, *with respect to the filtration generated by Z. If, in addition to i) and ii),* $\inf_{0\leq s\leq t} \kappa(Z(s)) > 0$ *a.s. for every* $t \geq 0$, *then, for every* $z \in \Xi \times \mathbb{R} \times \mathbb{R}$, *the process* $(M_2(t) : t \geq 0)$ *defined by*

$$M_2(t) = \kappa(Z(t)) \exp\left(-\int_0^t \frac{\beta(Z(s))}{\kappa(Z(s))} ds\right)$$

is also a local martingale, conditional on $Z(0) = z$, *with respect to the filtration generated by Z.*

Proof The proof that M_1 is a local martingale follows the lines of lemma 3.4, p. 176 in Ethier and Kurtz (1985). The fact that M_2 is a local martingale is a consequence of corollary 3.3, p. 66 of Ethier and Kurtz (1985). ■

We now are ready to rigorously verify the Equation (4). We shall assume, for the remainder of this section, that $\tilde{\lambda}$ and $\tilde{\gamma}$ are non-negative functions and that $\Gamma(t) \to \infty$ a.s. as $t \nearrow \infty$, given $Y(0) = y$ for any $y \in \Xi$. Gjessing and Paulsen (1997) provide rigorous verification for Equation (4) when the accumulated discount and reward processes follow independent Levy process; see also Pollack and Siegmund (1985) for the case in which Γ follows a geometric Brownian motion and Λ grows linearly.

Theorem 1 *Assume that*:

i) *For all* $y \in \Xi$, $\phi(y, \cdot)$ *is continuously differentiable on* $(-\infty, 0]$.
ii) *For all* $\theta \in (-\infty, 0]$, $\phi(\cdot, \theta) \in D(A_Y)$ *and* $\partial_\theta \phi(\cdot, \theta)$ *is continuous.*
iii) ϕ *is a bounded solution of* (4) *subject to* $\phi(\cdot, 0) = 1$.
iv) *The function* $\partial_\theta \phi(\cdot)$ *is bounded on compact subsets of* $\Xi \times (-\infty, 0]$.
v) *For each* $y \in \Xi$, $(Y(t) : t \geq 0)$ *is tight given* $Y(0) = y$.

Then $\phi(y, \theta) = E_y \exp(\theta D)$ *for* $(y, \theta) \in \Xi\times \trianglelefteq -\infty, 0]$.

Proof Letting $\kappa(Y(s), \Gamma(s)) = \phi(Y(s), \theta \exp(-\Gamma(s)))$, we can directly apply Lemma 1. In particular, note that

$$\begin{aligned}(A_Z\phi)\left(Y(s), \theta e^{-\Gamma(s)}\right) &= A\phi\left(Y(s), \theta e^{-\Gamma(s)}\right) - \theta\phi_\theta\left(Y(s), \theta e^{-\Gamma(s)}\right)e^{-\Gamma(s)}g(Y(s)) \\ &= -\theta e^{-\Gamma(s)}\tilde{\lambda}(Y(s))\phi\left(Y(s), \theta e^{-\Gamma(s)}\right).\end{aligned}$$

and therefore

$$M(t) = \phi\left(Y(t), \theta e^{-\Gamma(t)}\right)\exp\left(\theta\int_0^t e^{-\Gamma(s)}\tilde{\lambda}(Y(s))ds\right).$$

is a local martingale. In fact, since $M(\cdot)$ is bounded, the martingale convergence theorem guarantees that it converges almost surely and in L_1 as $t \nearrow \infty$. Let $\varepsilon > 0$ and pick K compact so that

$$P_y(Y(t) \in K) \geq 1 - \varepsilon.$$

We obtain that

$$\left|\phi(y, \theta) - E_y\left(\phi\left(Y(t)\theta e^{-\Gamma(t)}\right)\exp\left(\theta\int_0^t e^{-\Gamma(s)}\tilde{\lambda}(Y(s))ds\right); Y(t) \in K\right)\right| \leq c\varepsilon$$

for some $c > 0$. The conclusion of the result follows after showing that

$$\phi\left(Y(t), \theta e^{-\Gamma(t)}\right)I(Y(t) \in K) - I(Y(t) \in K) \to 0 \qquad (8)$$

as $t \nearrow \infty$. Equation (8) follows from the fact that

$$1 - \phi(y, \theta) = \int_\theta^0 \partial_\theta\phi(y, s)ds,$$

and because $\partial_\theta\phi(\cdot)$ is bounded on the compact set $K \times \theta, 0]$. ■

Sufficient conditions for existence/uniqueness to (4) have been well studied in the case in which Y is a diffusion with associated uniformly elliptic second-order differential operator L. In this case, we can transform Equation (4) into a parabolic equation by introducing the change of variable $\theta = -e^t$ (which imposes the restriction $\theta < 0$). We then let $\phi(y, e^t) = \tilde{\phi}(y, t)$ and find that (4) is transformed into

$$(A\tilde{\phi})(y, t) = \left(\partial_t\tilde{\phi}(y, t)g(y) - \tilde{\phi}(y, t)f(y)e^t\right), \quad (y, t) \in \Xi \times \mathbb{R}.$$

The verification of Equation (6) is given next.

Theorem 2 *Assume that*:

i) *For all* $y \in \Xi$, $h(y, \cdot)$ *is continuously differentiable on* $(-\infty, \infty)$.
ii) *For all* $z \in (-\infty, \infty)$, $h(\cdot, z) \in D(A_Y)$ *and* $\partial_z h(\cdot, z)$ *is continuous.*
iii) h is a bounded solution of (6).
iv) *Uniformly on compact sets of* Ξ, $h(\cdot, z) \to 1$ as $z \to \infty$ *and* $h(\cdot, z) \to 0$

as $z \to -\infty$

v) *For each* $y \in \Xi$, $(Y(t) : t \geq 0)$ *is tight given* $Y(0) = y$.

Then, $h(y, z) = P_y(D \leq z)$ *for each* $(y, z) \in \Xi \times \mathbb{R}$ *for which z is a continuity point of* $P_y(D \leq \cdot)$.

Proof Let

$$D(t) = \int_0^t e^{-\Gamma(s)} f(Y(s))ds,$$

and define $M = (M(t) : t \geq 0)$ as

$$M(t) = h\left(Y(t), (z - D(t))e^{\Gamma(t)}\right),$$

it follows from Lemma 1 that M is a local martingale. In fact, since h is bounded, M is martingale. Hence, it follows that

$$h(y, z) = E_y h\left(Y(t), (z - D(t))e^{\Gamma(t)}\right).$$

An argument similar to that given in the proof of Theorem 1 (using the uniform convergence on compact sets and the continuity of $P_y(D \leq \cdot)$ at z) implies, upon letting $t \to \infty$, that

$$h(y, z) = P_y(D \leq z)$$

as we claimed. ■

When $\tilde{\gamma}$ is bounded below by $\varepsilon > 0$ and $\tilde{\lambda}$ is non-negative and bounded (by m, say), then $0 \leq D \leq m/\varepsilon$, so that the boundary conditions for (6) can be modified to $h(y, 0) = 0$ and $h(y, m/\varepsilon) = 1$ for $y \in \Xi$ (so that the boundary conditions "at infinity" can be eliminated, simplifying numerical computation). We further note that for numerical computation of the distribution of D, solving (6) appears preferable to solving (4), followed by numerical inversion (given the similar degree of numerical difficulty associated with solving (4) and (6)).

Finally, we provide sufficient conditions for the verification of (5). We assume that $\tilde{\lambda}$ and $\tilde{\gamma}$ are non-negative and that $\Gamma(t) \to \infty$ as $t \nearrow \infty$ with probability one given $Y(0) = y$ for every $y \in \Xi$.

Theorem 3 *Assume that $\phi_0 = 1$ and for $1 \leq m \leq k$: Assume that $\phi_m(\cdot) \in A_Y$ for all $1 \leq m \leq k$, $\phi_0 = 1$ and that (5) is satisfied. In addition, suppose that $\phi_m(\cdot)$ is bounded. Then, $\phi_k(y) = E_y D^k$.*

Proof The proof proceeds by induction along the lines of the previous results by studying the Dynkin martingale generated by $\kappa_1(Y(t), \Gamma(t)) = \phi_1(Y(t))\exp(-\Gamma(t))$ and subsequently to $\kappa_m(Y(t), \Gamma(t)) = \phi_m(Y(t))\exp(-m\Gamma(t))$ for $m \leq k$. The details are omitted. ■

3 | WEAK CONVERGENCE

Our results in this section in particular will imply that when the accumulated force of interest process $\Gamma(\cdot)$ is close to zero, then we can approximate the distribution of D as

$$D \overset{D}{\approx} N(ED, Var(D)), \tag{9}$$

where $N\left(\mu, \sigma^2\right)$ is a Gaussian rv having mean μ and variance σ^2, and $\overset{D}{\approx}$ is a non-rigorous symbol meaning "has approximately the same distribution as." The mean ED and $Var(D)$ can either be computed by solving the first two moment equations of Section 2 (which are substantially easier to solve numerically than computing the entire distribution of D) or by using the approximations to ED and $Var(D)$ that appear in the limit theorem given below. Our limit theorem below also supports other types of approximations (based on stable processes) depending on the structure of the rewards or the interest rate dynamics.

There are other results in the literature that suggest Gaussian limit laws for D (see for instance Nelson (1990), Bucklew et al. (1993) and Forniari and Mele (1997)), where weak convergence proofs are provided at the process level on finite time intervals. Of course, it is easy to construct examples for which weak convergence fails for the infinite horizon perpetuity D despite the process-level weak convergence on finite time intervals. A major contribution of the results in this section is to provide general machinery for extending finite-time weak convergence results to the infinite horizon quantity D. Moreover, as we have mentioned before, our assumptions do not require that the accumulated force of interest and rewards satisfy a Brownian weak convergence law, nor do we require strong assumptions on the discounting process as in Whitt (1972) or Gerber (1971) (where the force of interest is assumed to be constant). Finally, in contrast to most of the prior work discussed earlier, our results are developed both in discrete and continuous time, which is a feature that is convenient in some of the applications discussed in Section 1, in particular in the context of insurance and finance.

To make the above approximation rigorous, we introduce a parameter α and a family of systems indexed by α in the following way. Assume that the cumulative force of interest and the accumulated reward process the form $(\alpha\Gamma_\alpha(t) : t \geq 0)$ and $(\Lambda_\alpha(t) : t \geq 0)$, respectively. The idea is to now study the rv

$$D(\alpha) = \int_0^\infty \exp\left(-\alpha\Gamma_\alpha\left(t_-\right)\right) d\Lambda_\alpha(t) \tag{10}$$

as $\alpha \searrow 0$. As noted earlier, a small force of interest is often a reasonable assumption in the setting of insurance perpetuities. In the ARCH contest, α small corresponds to a model that is close to a unit autoregressive root (when γ is deterministic). Consequently, the asymptotic regime in which $\alpha \searrow 0$ is of significant applied interest when computing approximations to the distribution of D.

We prove here a very general central limit theorem (CLT) for D. In order to make sense of the stochastic integral appearing in (10), we assume that Λ is a semi-martingale adapted to a suitable filtration, so that

$$\Lambda_\alpha = \Lambda_\alpha^{(b)} + \Lambda_\alpha^{(M)},$$

where $\Lambda_\alpha^{(M)}$ is a local martingale and $\Lambda_\alpha^{(b)}$ is a process with locally bounded variation (i.e., bounded variation on finite time intervals). We further require that Λ_α and Γ_α be cadlag processes, with Γ_α being adapted to the same filtration as Λ_α.

Our first assumption is a "finite time" functional limit theorem for Λ_α and Γ_α.

W1 Assume that there exist constants $\gamma > 0$, $\lambda \in (-\infty, \infty)$ and $\beta \in (0, 1]$ such that

$$\left(\overline{\Gamma}_\alpha, \overline{\Lambda}_\alpha\right) \triangleq \alpha^{-\beta}\left(\alpha\Gamma_\alpha(\cdot/\alpha) - \gamma\cdot, \alpha\Lambda_\alpha(\cdot/\alpha) - \lambda\cdot\right) \Rightarrow (Z_\Gamma(\cdot), Z_\Lambda(\cdot))$$

as $\alpha \searrow 0$ in $D^2[0,\infty) = D[0,\infty) \times D[0,\infty)$ (under the standard Skorohod J_1 topology).

Note that in typical applications, (Z_Γ, Z_Λ) will be a two-dimensional Brownian motion with $\beta = 1/2$. But W1 permits more general limit processes, like stable process and fractional stable motions (as might be appropriate for models containing heavy-tailed rv's). In such applications, it may be that the natural (marginal) normalizations for each of the processes $\alpha\Gamma_\alpha(\cdot/\alpha)$ and $\alpha\Lambda_\alpha(\cdot/\alpha)$ are different, say $\alpha^{-\beta_1}$ and $\alpha^{-\beta_2}$, respectively. In such contexts, we set $\beta = \min(\beta_1, \beta_2) > 0$ in W1, so that one of the two limit processes is degenerate and identically equal to zero.

In the proof of the central limit theorem, it is necessary to deal with weak convergence of a stochastic integral that involves integrating $\Gamma_\alpha(\cdot)$ against $\Lambda_\alpha(\cdot)$. The next condition is closely related to a key condition identified by Kurtz and Protter (1991) that justifies weak convergence of stochastic integrals based on the joint weak convergence of the integrand and integrator.

W2 There exists $\alpha_0 > 0$ such that

$$\overline{\lim_{t\to\infty}} \sup_{0<\alpha\le\alpha_0}\left(\frac{E\left|\Lambda_\alpha^{(b)}\right|(t)}{t} + \frac{E\left[\Lambda_\alpha^{(M)}\right](t)}{t}\right) < \infty,$$

where $\left[\Lambda_\alpha^{(M)}\right](\cdot)$ denotes the quadratic variation of $\Lambda_\alpha^{(M)}(\cdot)$ and $\left|\Lambda_\alpha^{(b)}\right|(\cdot)$ is the total variation of $\Lambda_\alpha^{(b)}(\cdot)$.

As noted earlier, because W1 does not control the finite-time behavior of the process, it is easily seen that W1, by itself, does not guarantee a limit law for $D(\alpha)$ as $\alpha \searrow 0$. Our final condition imposes the necessary regularity needed to control the discounted infinite-time behavior of the process underlying the perpetuity.

W3

$$\overline{\lim_{\alpha\searrow 0}} E \log\left(1 + \sup_{u\in 0,1]}\left|\overline{\Lambda}_\alpha(u)\right|\right) < \infty,$$

$$\overline{\lim_{\alpha\searrow 0}} E\left(\sup_{u\in 0,1]}\left|\overline{\Gamma}_\alpha(u)\right|\right) < \infty.$$

We are now ready to state the main result of this section.

Theorem 4 *Under W1 to W3,*

$$\alpha^{-\beta}(D(\alpha) - \lambda/\gamma) \Rightarrow \int_0^\infty e^{-\gamma s} dZ_1^\Lambda(s) - \lambda\int_0^\infty e^{-\gamma s} dZ_2^\Gamma(s)$$

as $\alpha \searrow 0$.

Remark The stochastic integrals appearing in the limiting rv can be equivalently expressed, via an appropriate integration by parts, as

$$\int_0^\infty e^{-\gamma s} dZ_1^\Lambda(s) - \lambda\int_0^\infty e^{-\gamma s} dZ_2^\Gamma(s) = \gamma\int_0^\infty e^{-\gamma s} Z_1^\Lambda(s)ds - Z_1^\Lambda(0) - \lambda\gamma\int_0^\infty e^{-\gamma s} Z_2^\Gamma(s)ds + \lambda Z_2^\Gamma(0).$$

The proof also shows that the resulting rv's are finite-valued quantities under W1 to W3.

Remark Condition W2 can be somewhat relaxed, by means of localization, as in Kurtz and Protter (1991), condition C2.2(i).

The following lemma connects condition W3 to a more suitable technical condition that controls the large-time behavior of the processes Λ and Γ. The proof of this lemma is given at the end of the present section.

Lemma 2 *If condition* ***W3*** *holds, then for each* $\delta, \delta_0 > 0$ *we have that*

$$\overline{\lim_{t_0\nearrow\infty}}\, \overline{\lim_{\alpha\searrow 0}} P\left(\sup_{t\ge t_0} e^{-\delta t}\left|\overline{\Lambda}_\alpha(u)\right| > \delta_0\right) = 0.$$

Moreover,

$$\overline{\lim_{t_0\nearrow\infty}}\, \overline{\lim_{\alpha\searrow 0}} P\left(\sup_{t\ge t_0}\left|\overline{\Gamma}_\alpha(u)\right| > \delta_0\right) = 0.$$

With the aid of the previous lemma we can proceed to the proof of the main result of this section.

Proof of Theorem 4 Note that

$$\begin{aligned}\alpha^{-\beta}&\left(\int_0^t e^{-\alpha\Gamma_\alpha(s/\alpha-)}\alpha d\Lambda_\alpha(s/\alpha) - \int_0^t e^{-\gamma s}\lambda ds\right)\\ &= \alpha^{-\beta}\int_0^t \left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha(s/\alpha) \\ &\quad + \alpha^{-\beta}\int_0^t e^{-\gamma s} d\left[\alpha\Lambda_\alpha(s/\alpha) - \lambda s\right].\end{aligned} \tag{11}$$

Now observe that

$$\alpha\Lambda_\alpha(\cdot/\alpha) \Rightarrow \lambda\cdot$$

as $\alpha \searrow 0$ in $D[0,\infty)$ and because the limit is deterministic and continuous the convergence actually holds in probability and uniformly on compact sets. Now, we wish to show that

$$\begin{aligned}&\left(\int_0^\cdot e^{-\gamma s} d\overline{\Lambda}_\alpha(s), \alpha^{-\beta}\int_0^\cdot \left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha(s/\alpha), \overline{\Gamma}_\alpha(\cdot), \overline{\Lambda}_\alpha(\cdot)\right)\\ &\overset{D^4[0,\infty)}{\Longrightarrow} \left(\int_0^\cdot e^{-\gamma s} dZ_1^\Lambda(s), -\lambda\int_0^\cdot e^{-\gamma s} Z_2^\Gamma(s)ds, Z_2^\Gamma(\cdot), Z_1^\Lambda(\cdot)\right).\end{aligned} \tag{12}$$

This result is expected given assumption W1 and the fact that

$$e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s} = e^{-\gamma s}\left(e^{-[\alpha\Gamma_\alpha(s/\alpha-)-\gamma s]} - 1\right)$$
$$\approx -e^{-\gamma s}\left(\alpha\Gamma\left(s/\alpha_-\right) - \gamma s\right),$$

which suggests the convergence of the second component in (12). Now, in order to justify the convergence in (12), we will apply theorem 7.10 of Kurtz and Protter (1996). In particular, such a result guarantees that if the integrand and integrator converge weakly jointly and the integrator possesses "uniformly controlled variations" (UCV), according to definition 7.5 of Kurtz and Protter (1996), then weak convergence in the form of (12) is justified. So, in order to deal with the second component of the vector process in (12) we need to ensure UCV of $\alpha\Lambda_\alpha(\cdot/\alpha)$. Equation (7.12) of Kurtz and Protter (1996) indicates that UCV is satisfied if we can show that for any $u > 0$

$$\sup_{0<\alpha\leq\alpha_0}\left(E\left|\alpha\Lambda_\alpha^{(b)}\right|(u/\alpha) + \alpha^2 E\left[\Lambda_\alpha^{(M)}\right](u/\alpha)\right) < \infty.$$

This condition is readily satisfied in view of W2. Furthermore, a simple Taylor development implies

$$\left|\alpha^{-\beta}e^{-\gamma s}\left[e^{-(\alpha\Gamma_\alpha(s/\alpha-)-\gamma s)} - 1\right] - \alpha^{-\beta}e^{-\gamma s}\left[\alpha\Gamma_\alpha(s/\alpha-) - \gamma s\right]\right|$$
$$\leq \alpha^{-\beta}e^{-\gamma s}|\alpha\Gamma_\alpha(s/\alpha-) - \gamma s|^2 e^{|\alpha\Gamma_\alpha(s/\alpha-)-\gamma s|}$$
$$\leq \alpha^{\beta}e^{-\gamma s}\alpha^{-2\beta}|\alpha\Gamma_\alpha(s/\alpha-) - \gamma s|^2 e^{|\alpha\Gamma_\alpha(s/\alpha-)-\gamma s|}.$$

Sending $\alpha \searrow 0$ and using assumption W1 we then obtain that

$$\left|\alpha^{-\beta}e^{-\gamma s}\left[e^{-(\alpha\Gamma_\alpha(s/\alpha-)-\gamma s)} - 1\right] + \alpha^{-\beta}e^{-\gamma s}\left[\alpha\Gamma_\alpha(s/\alpha-) - \gamma s\right]\right| \to 0$$

in probability uniformly on compact sets and therefore we can invoke theorem 7.10 of Kurtz and Protter (1996) to conclude

$$\left(\alpha^{-\beta}\int_0^\cdot\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha(s/\alpha), \overline{\Gamma}_\alpha(\cdot), \overline{\Lambda}_\alpha(\cdot)\right)$$
$$\overset{D^3[0,\infty)}{\Longrightarrow}\left(-\lambda\int_0^\cdot e^{-\gamma s}Z_2^\Gamma(s)ds, Z_2^\Gamma(\cdot), Z_1^\Lambda(\cdot)\right).$$

In order to lift the previous convergence result to the four components in (12) we use lemma 2.1 of Whitt (1972) to show that the first component of (12) is a continuous function of $\overline{\Lambda}_\alpha(\cdot)$ in $D[0,\infty)$. Now we can apply the continuous mapping principle here to the sum of the first two components of (12) in $D^4[0,\infty)$, the validity of this application is ensured because $\int_0^\cdot e^{-\gamma s}Z_2^\Gamma(s)ds$ is continuous (see Whitt (2001)), yielding

$$\alpha^{-\beta}\left(\int_0^\cdot e^{-\alpha\Gamma_\alpha(s/\alpha-)}\alpha d\Lambda_\alpha(s/\alpha) - \int_0^\cdot e^{-\gamma s}\lambda ds\right)$$
$$\overset{D[0,\infty)}{\Longrightarrow}\int_0^\cdot e^{-\gamma s}dZ_1^\Lambda(s) - \lambda\int_0^\cdot e^{-\gamma s}Z_2^\Gamma(s)ds.$$

Now, to extend the previous result to the infinite horizon and complete the proof, we must show that for each $\delta_0, \varepsilon > 0$, there exists $t = t(\varepsilon) > 0$ large enough such that

$$\overline{\lim_{\alpha\to 0}} P\left(\left|\alpha^{-\beta}\left(\int_t^\infty e^{-\alpha\Gamma_\alpha(s/\alpha-)}\alpha d\Lambda_\alpha(s/\alpha) - \int_t^\infty e^{-\gamma s}\lambda ds\right)\right| > \delta_0\right) \leq \varepsilon. \tag{13}$$

So, we have to study

$$\alpha^{-\beta}\left(\int_t^\infty e^{-\alpha\Gamma_\alpha(s/\alpha-)}\alpha d\Lambda_\alpha(s/\alpha) - \int_t^\infty e^{-\gamma s}\lambda ds\right)$$
$$= \alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha(s/\alpha) \tag{14}$$
$$+ \alpha^{-\beta}\int_t^\infty e^{-\gamma s}d\left(\alpha\Lambda_\alpha(s/\alpha) - \lambda s\right). \tag{15}$$

First we analyze (14). We decompose this term using the decomposition for the semi-martingale Λ_α. We find that the integral (14) equals

$$\alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(b)}(s/\alpha) \tag{16}$$
$$+ \alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(M)}(s/\alpha). \tag{17}$$

We shall start by showing that the contribution of the integral (17) is small for large t (uniformly in α). Define the stopping time

$$T_1 = \inf\left\{u \geq t : u^{-1}\left|\alpha\Gamma_\alpha(u/\alpha) - \gamma u\right| > \delta_1\alpha^\beta\right\},$$

Note that

$$P\left(\left|\alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(M)}(s/\alpha)\right| > \delta\right) \tag{18}$$
$$\leq P\left(\left|\alpha^{-\beta}\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(M)}(s/\alpha)\right| > \delta; T_1 = \infty\right)$$
$$+ P\left(T_1 < \infty\right)$$
$$\leq \left(\alpha^{-2(\beta-1/2)}/\delta\right)E\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)} - e^{-\gamma s}\right)^2\alpha d\left[\Lambda_\alpha^{(M)}\right](s/\alpha) \tag{19}$$
$$+ P\left(T_1 < \infty\right). \tag{20}$$

Now, observe that, for some constant $c_1 > 0$,

$$E\int_t^{T_1} e^{-2\gamma s}\left(e^{-(\alpha\Gamma_\alpha(s/\alpha-)-\gamma s)} - 1\right)^2\alpha d\left[\Lambda_\alpha^{(M)}\right](s/\alpha)$$

$$\leq c_1\alpha^\beta E\int_t^{T_1} e^{-\gamma s}\alpha d\left[\Lambda_\alpha^{(M)}\right](s/\alpha)$$
$$\leq c_1\alpha^\beta E\int_t^{\infty} e^{-\gamma s}\alpha d\left[\Lambda_\alpha^{(M)}\right](s/\alpha).$$

This estimate implies that (19) is bounded by

$$c_1\frac{\alpha^{-2(\beta-1/2)}}{\delta}\alpha^\beta E\int_t^\infty e^{-\gamma s}\alpha d\left[\Lambda_M\right](s/\alpha)$$
$$\leq c_1\gamma\frac{\alpha^{1-\beta}}{\delta}E\int_t^\infty e^{-\gamma s}\alpha\left[\Lambda_\alpha^{(M)}\right](s/\alpha)ds < c_2\alpha^{1-\beta}e^{-t\gamma/2}, \quad (21)$$

for some constant $c_2 > 0$ (where we have used assumption W2 combined with integration by parts). For $P(T_1 < \infty)$, note that

$$P(T_1<\infty)\leq P\left(\sup_{u\geq t}\left|\frac{\alpha\Gamma_\alpha(u/\alpha)-\gamma u}{u}\right|\geq\delta_1\alpha^\beta\right).$$

Therefore, this estimate combined with (21) and Lemma 2 yields that for each $\varepsilon > 0$, we can find $t = t(\varepsilon)$ sufficiently large and $\alpha_0 > 0$ small enough such that

$$\sup_{0<\alpha<\alpha_0} P\left(\left|\alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s/\alpha-)}-e^{-\gamma s}\right)d\Lambda_\alpha^{(M)}(s/\alpha)\right|>\delta\right)$$
$$\leq\varepsilon. \quad (22)$$

This takes care of (18). The analysis of (16) is similar to that of (17). In particular, we note that

$$P\left(\left|\alpha^{-\beta}\int_t^\infty\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(b)}(s/\alpha)\right|>\delta\right)$$
$$\leq P\left(\left|\alpha^{-\beta}\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(b)}(s/\alpha)\right|>\delta\right)$$
$$+P(T_1<\infty). \quad (23)$$

Note that the definition of T_1 implies that we can find a constant $c_1 > 0$ and a finite random variable $C(\omega)$ such that

$$P\left(\left|\alpha^{-\beta}\int_t^{T_1}\left(e^{-\alpha\Gamma_\alpha(s-)}-e^{-\gamma s}\right)\alpha d\Lambda_\alpha^{(b)}(s/\alpha)\right|>\delta\right)$$
$$\leq P\left(c_1\int_t^{T_1}e^{-\gamma s/2}\alpha d\left|\Lambda_\alpha^{(b)}\right|(s/\alpha)>\delta\right)$$
$$\leq P\left(c_1\int_t^{\infty}e^{-\gamma s/2}\alpha d\left|\Lambda_\alpha^{(b)}\right|(s/\alpha)>\delta\right)$$
$$\leq P\left(C(\omega)e^{-\gamma t/2}>\delta\right), \quad (24)$$

where the last line above follows from integration by parts and assumption W2. Consequently, (24) together with (22) allows one to control the behavior of (14). Finally, we study (15). Integration by parts yields

$$\alpha^{-\beta}\int_t^\infty e^{-\gamma s}d\left[\alpha\Lambda_\alpha(s/\alpha)-\lambda s\right]$$
$$=\alpha^{-\beta}\int_t^\infty\gamma\alpha\Lambda_\alpha(s/\alpha)-\lambda s]e^{-\gamma s}ds-e^{-\gamma t}\alpha^{-\beta}\left[\alpha\Lambda_\alpha(s/\alpha)-\lambda s\right].$$

Hence, as an immediate consequence of Lemma 2, we obtain that

$$\overline{\lim_{t\nearrow\infty}}\,\overline{\lim_{\alpha\searrow 0}}\,P\left(\alpha^{-\beta}\left|\int_t^\infty e^{-\gamma s}d\left[\alpha\Lambda_\alpha(s/\alpha)-\lambda s\right]\right|>\delta_0\right)=0,$$

yielding (13). ∎

Proof of Lemma 2 Let t_0 be a large but fixed number, $\alpha > 0$ and pick $\theta > 1$. Then

$$P\left(\sup_{t\geq t_0}e^{-\delta t}\alpha^{-\beta}\left|\Lambda_\alpha(t/\alpha)-\lambda t\right|>\delta_0\right)$$
$$\leq\sum_{k=0}^\infty P\left(\sup_{t\in t_0\theta^k,t_0\theta^{k+1})}e^{-\delta t}\alpha^{-\beta}\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|>\delta_0\right)$$
$$\leq\sum_{k=0}^\infty P\left(\sup_{t\in t_0\theta^k,t_0\theta^{k+1})}\alpha^{-\beta}\frac{\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|}{\left(t_0\theta^{k+1}\right)^\beta}>\frac{t_0\exp\left(\delta\theta^k\right)\delta_0}{\left(t_0\theta^{k+1}\right)^\beta}\right)$$
$$\leq\sum_{k=0}^\infty P\left(\sup_{t\in 0,t_0\theta^{k+1})}\alpha^{-\beta}\frac{\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|}{\left(t_0\theta^{k+1}\right)^{1-\beta}}>\frac{t_0^\beta\exp\left(\delta\theta^k\right)\delta_0}{\theta^{(k+1)\beta}}\right). \quad (25)$$

Put $r_k = t_0\theta^{k+1}$, and set $ur_k = t$ and $\alpha/r_k = \widetilde{\alpha}$. Then,

$$\sup_{t\in 0,r_k]}\alpha^{-\beta}\frac{\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|}{r_k^{1-\beta}}=\sup_{u\in 0,1]}\frac{\left|\alpha\Lambda_\alpha\left(ur_k/\alpha\right)-\lambda ur_k\right|}{\alpha^\beta r_k^{1-\beta}}$$
$$=\sup_{u\in 0,1]}\frac{\left|\alpha\Lambda_\alpha\left(ur_k/\alpha\right)-\lambda ur_k\right|}{(\alpha/r_k)^\beta r_k}$$
$$=\sup_{u\in 0,1]}\frac{\left|\alpha\Lambda_\alpha\left(ur_k/\alpha\right)-\lambda ur_k\right|}{\widetilde{\alpha}^\beta}.$$

Therefore, by property W3, (25) implies that there exists a constant $b > 0$ such that for all α sufficiently small

$$P\left(\sup_{t\geq t_0}e^{-\delta t}\alpha^{-\beta}\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|>\delta_0\right)$$
$$\leq\sum_{k=0}^\infty P\left(\sup_{t\in 0,t_0\theta^{k+1})}\alpha^{-\beta}\frac{\left|\alpha\Lambda_\alpha(t/\alpha)-\lambda t\right|}{\left(t_0\theta^{k+1}\right)^{1-\beta}}>\frac{t_0^\beta\exp\left(\delta\theta^k\right)\delta_0}{\theta^{(k+1)\beta}}\right)$$
$$\leq b\sum_{k=0}^\infty\frac{1}{\beta\log(t_0)+\delta\theta^k-(k+1)\beta\log(\theta)-\log\delta_0}.$$

Since $\theta > 1$, the previous quantity is finite and it goes to zero (because $\beta > 0$) as $t_0 \nearrow \infty$. The corresponding property for Γ_α follows completely analogous steps and therefore is omitted. This concludes the proof of the lemma. ∎

Now, let us discuss how the previous result can be applied in practice. In many applications, the processes $(\Gamma_\alpha, \Lambda_\alpha)$ can be taken to be independent of α. In addition, one often has $\beta = 1/2$ and Z corresponds to Brownian motion, so that

$$\alpha^{1/2}\left(\begin{pmatrix}\Lambda(\cdot/\alpha)\\ \Gamma(\cdot/\alpha)\end{pmatrix}-\begin{pmatrix}\lambda\cdot/\alpha\\ \gamma\cdot/\alpha\end{pmatrix}\right)\Rightarrow G\begin{pmatrix}B_1(\cdot)\\ B_2(\cdot)\end{pmatrix},$$

where $B = (B_1, B_2)$ is a two dimensional Brownian motion and $GG^T = C$ is the corresponding covariance matrix. Typically, one would have

$$C_{1,2} = \lim_{t\to\infty} E\frac{(\Lambda(t) - \lambda t)(\Gamma(t) - \gamma t)}{t},$$

$$C_{1,1} = \lim_{t\to\infty} E\frac{(\Lambda(t) - \lambda t)^2}{t},$$

$$C_{2,2} = \lim_{t\to\infty} E\frac{(\Gamma(t) - \gamma t)^2}{t}.$$

Therefore, Theorem 4 guarantees that under mild assumptions

$$\alpha^{-1/2}(\alpha D(\alpha) - \lambda/\gamma) \Rightarrow Z(\infty) \overset{D}{=} \sigma/\gamma^{1/2} N(0,1),$$

where

$$\sigma^2 = \frac{1}{2}\left(C_{11} - 2\frac{\lambda}{\gamma}C_{12} + \frac{\lambda^2}{\gamma^2}C_{22}\right).$$

The expression for σ^2 was obtained using integration by parts, which allows us to obtain the representation

$$Z(\infty) = \int_0^\infty e^{-\gamma s} d\left(G_1.B(s)\right) - \frac{\lambda}{\gamma}\int_0^\infty e^{-\gamma s} d\left(G_2.B(s)\right),$$

where $G_{1\cdot} = \left(G_{11}, G_{1,2}\right)$ and $G_{2\cdot} = (G_{21}, G_{22})$. This representation, combined with Ito's isometry, yields σ^2. In practical situations, one is typically dealing with a single problem instance, so that the parameter $\alpha > 0$ does not appear naturally in the problem structure. Theorem 4 then provides rigorous support for the formal approximation

$$D = \int_0^\infty \exp\left(-\Gamma\left(t_-\right)\right) d\Lambda(t) \overset{D}{\approx} \frac{\lambda}{\gamma} + \sigma/\gamma^{1/2} N(0,1), \quad (26)$$

which is free of α. Under appropriate uniform integrability in Theorem 4, $ED(\alpha) = \lambda/\gamma\alpha + o\left(\alpha^{-1/2}\right)$ and $Var(D(\alpha)) = \sigma^2/\gamma\alpha + o(1)$ as $\alpha \searrow 0$. We can therefore interpret Theorem 4 as also offering rigorous support for (9) in the presence of small interest rates. Because (9) uses the exact moments, we expect a better approximation to the distribution of D than as in the case of (26). On the other hand, (26) does not require solving any equations, whereas (9) requires computing ϕ_1 and ϕ_2..

4 | EDGEWORTH EXPANSIONS

In this section, we provide refined versions for some of the approximations given in the previous sections. The refined approximation takes the form of an Edgeworth expansion for the distribution of D. We shall derive these approximations in the iid setting for the discrete time case and under Markovian assumptions for the continuous time case. More precisely, in the discrete time case, motivated by the applications to ARCH processes described in Section 2, we consider

$$D = \sum_{k=0}^{\infty} \exp\left(-\sum_{j=0}^{k-1} Z_j\right) X_k,$$

where $(X_k, Z_k)_{k\geq 1}$ is a sequence of iid random vectors satisfying certain assumptions to be described later (see assumptions ED1 to ED4 below); while in the continuous time context, we work with

$$D = \int_0^\infty \exp\left(-\int_0^t \gamma(Y(s))ds\right) d\Lambda(t),$$

where $Y = (Y(s) : s \geq 0)$ is a time-homogeneous Markov process and Λ is a stationary independent increment process. This pair of assumptions is motivated by our interest in risk theory applications. The stationary independent increment assumption of the risk process Λ has been argued to hold by many authors in the risk theory community and includes the so-called classical risk model; see Asmussen (2001) and Grandell (1991)). Gjessing and Paulsen (1997) assume that Γ and Λ are two independent Levy processes. Our development here replaces the Levy assumption on Γ with the more common finance assumption that the short-rate process can be modeled as an ergodic diffusion with mean reversion.

Before presenting our mathematical development, let us discuss the general strategy that we will follow. The basic idea (say, assuming just momentarily that the X_k's are positive or that Λ is increasing so that $ED > 0$) is to develop an expansion for the complex cumulant generating function $\log E\exp\left(i\theta(ED)^{-1/2}(D - ED)\right)$. This approach is common in the development of Gram-Charlier and Edgeworth expansions. The development of the approximation proceeds by Fourier inversion. In particular, we obtain

$$E\exp\left(i\theta(ED)^{-1/2}(D - ED)\right) \approx \exp\left(-\frac{\theta^2 Var(D)}{2ED}\right)\left(1 - \frac{i\theta^3\kappa_{\underline{D}}^{(3)}}{3!}\right), \quad (27)$$

where $\kappa_{\underline{D}}^{(3)}$ corresponds to the third cumulant of the centered and scaled random variable $(D - ED)/(ED)^{1/2}$. A sensible approximation then becomes

$$P\left((ED)^{-1/2}(D - ED) \leq x\right) \approx P(N(0, Var(D)/ED) \leq x) - (ED)^{3/2}\frac{\kappa_{\underline{D}}^{(3)} H\left(x(ED/Var(D))^{1/2}\right)}{3!Var(D)^{3/2}}, \quad (28)$$

where $H(y) = \left(y^2 - 1\right)\eta(y)$ and $\eta(y) = \exp\left(-y^2/2\right)/(2\pi)^{1/2}$.

The formal approximation (28) will be rigorously justified, as in the previous section, in the context of small discount rates. We shall study, for $\alpha > 0$ small

$$D(\alpha) = \sum_{k=0}^{\infty} \exp\left(-\alpha\sum_{j=0}^{k-1} Z_j\right) X_k.$$

The strategy is to analyze $\psi_\alpha(\theta) \triangleq \log E\exp\left(i\theta\alpha^{1/2}(D(\alpha) - \lambda/(\alpha\gamma))\right)$. Note that in this setting we typically have (under the appropriate uniform integrability assumptions) $\alpha ED(\alpha) \approx \lambda/\gamma$ and therefore $\psi_\alpha(\theta)$ corresponds asymptotically to the cumulant generating function displayed of $(D - ED)/(ED)^{1/2}$. One can write

$$\exp(\psi_\alpha(\theta)) = \exp\left(v_1(\alpha)i\theta\alpha^{1/2} - v_2(\alpha)\theta^2\alpha/2 - i\theta^3\alpha^{3/2}v_3(\alpha)/3! + \cdots\right),$$

where $v_k(\alpha)$ is the kth cumulant of $D(\alpha) - \lambda(\alpha\gamma)^{-1}$. It is natural to expect that $v_1(\alpha) = c_1 + o(1)$, $v_2(\alpha) = \sigma^2\alpha^{-1} + c_2 + o(1)$, $v_3(\alpha) = c_3\alpha^{-1} + O(1)$, which would yield (up to quantities of order $o\left(\alpha^{1/2}\right)$)

$$\exp(\psi_\alpha(\theta)) \approx \exp\left(-\sigma^2\theta^2/2\right)\left(1 + \left(c_1 i\theta + c_3(i\theta)^3\right)\alpha^{1/2}\right). \tag{29}$$

This expression is in correspondence (neglecting terms of order $o\left(\alpha^{1/2}\right)$) with (27). A formal Edgeworth expansion is then obtained by applying inverse Fourier transforms to both sides of the previous approximation (as previously indicated for (28)). The following subsections provide rigorous support to the Edgeworth expansion obtained after sending $\alpha \searrow 0$; we also provide explicit expressions for the c_k's. We shall carry over this program both in the discrete-time iid case and the continuous-time setting under Markovian assumptions.

4.1 | The discrete-time setting

In this section, we shall consider the following set of assumptions.

ED1 Assume that $Z_1 \geq 0$, $E(Z_1) = \gamma < \infty$, $E\left(Z_1^2\right) = \mu_Z^{(2)} < \infty$, and $E\left(|Z_1|^3\right) < \infty$. Let σ_Z^2 be the variance of Z_1 and $\kappa_Z^{(3)}$ its third order cumulant, which can be written as

$$\kappa_Z^{(3)} = \mu_Z^{(3)} - 3\mu_Z^{(2)}\gamma + 2\gamma^3.$$

ED2 Suppose that X_1 has non-lattice distribution with $E(X_1) = \lambda$, $Var\left(X_1^2\right) = \sigma_X^2$, and $E\left(|X_1|^3\right) < \infty$. Let $E\left(X_1^3\right) = \mu_3^X$ and write $\kappa_X^{(3)}$ to denote the third order cumulant of X_1.

ED3 Suppose that $E\left(|X_1|^j|Z_1|^k\right) < \infty$ for $0 < j+k \leq 3$ and for $j, k \geq 1$ denote $\mu_{jk} = E\left(X_1^j Z_1^k\right)$. Moreover, let us define,

$$\delta(\theta, Z_1) = \left|E\left(e^{i\theta X_1}\,\middle|\, Z_1\right)\right|$$

and assume that

$$\overline{\lim_{h\to 0}} \sup_{\varepsilon \leq |\theta| \leq 1/\varepsilon} \frac{P(\delta(\theta, Z_1) > 1 - h)}{h} < \infty, \tag{30}$$

for $\varepsilon > 0$.

Condition (30) may be seen as a form of strong non-latticity of X_1 given Z_1. Notice that in the important special case in which the X_k's are independent of the Z_k's, assumption ED3 is an immediate consequence of ED2. Indeed, if X_1 is non-lattice, we have that $\delta(\theta, Z_1) = \delta(\theta) < 1$. Therefore, for all $h > 0$ sufficiently small, $\delta(\theta) < 1 - h$. This implies that the limit in (30) is zero.

As a remark, we also note that, alternatively, the non-negativity of Z_1 required in assumption ED1 can be replaced by the existence of exponential moments and $EZ_1 > 0$. We record this observation as our alternative assumption ED1'.

ED1′ Assume that $EZ_1 > 0$ and $E\exp(\rho Z_1) < \infty$ for ρ in a vicinity of the origin.

Under these assumptions, the constants σ^2, c_1, and c_2 appearing in (29) take the form

$$\sigma^2 = \frac{1}{2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right),$$

$$c_1 = \frac{\mu_Z^{(2)}\lambda}{2\gamma^2},$$

$$c_3 = \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} - 3\frac{\kappa_{11}}{\gamma}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) + 3\sigma_Z^2\frac{\lambda}{\gamma^2}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3},$$

with

$$\kappa_{12} = \mu_{12} + \mu_{11} - \mu_Z^{(2)} - 3\gamma\mu_{11} + 2\gamma^2\lambda,$$

$$\kappa_{21} = \mu_{21} + \mu_{11} - \mu_X^{(2)} - 3\lambda\mu_{11} + 2\lambda^2\gamma,$$

$$\kappa_{11} = \mu_{11} - \lambda\gamma = \sigma_{XZ} \triangleq cov(X, Z).$$

In view of (28), this analysis yields the approximation,

$$P(D \leq y) \approx P\left(N\left(\lambda/\gamma, \sigma^2/\gamma\right) \leq y\right) - \sqrt{\gamma}\beta_1\eta\left((y - \lambda/\gamma)\frac{\sqrt{\gamma}}{\sigma}\right) - \frac{\sqrt{\gamma}}{18}\beta_2 H\left((y - \lambda/\gamma)\frac{\sqrt{\gamma}}{\sigma}\right), \tag{31}$$

where $\beta_1 = c_1/\sigma$ and $\beta_2 = c_3/\sigma^3$, and $\eta(y)$ and $H(y)$ were defined after display (28). In approximation (31), the exact moments appearing in (26) have been replaced by their asymptotic limits (as $\alpha \to 0$). This Edgeworth approximation can be viewed as a refinement of the normal approximation (26).

To rigorously state our Edgeworth approximation, we introduce (as we done before) a small scaling parameter $\alpha > 0$ and define

$$D(\alpha) = \sum_{k=0}^{\infty} \exp\left(-\alpha\sum_{j=0}^{k-1} Z_j\right) X_k.$$

Theorem 5 *If the set of assumptions ED1 to ED3 are in force (or if ED1′, ED2, and ED3 hold), then*

$$P\left(\sqrt{\alpha}\left(D(\alpha) - \frac{\lambda}{\gamma\alpha}\right) \leq y\right) = P\left(N\left(0, \frac{\sigma^2}{\gamma}\right) \leq y\right) - \sqrt{\alpha}\beta_1\eta(y) - \frac{\sqrt{\alpha}}{18}\frac{\beta_2}{\gamma}H(y) + G_\alpha(y); \tag{32}$$

where $G_\alpha = o(\sqrt{\alpha})$ uniformly over y in compact intervals as $\alpha \searrow 0$.

Before presenting the proof of Theorem 1, we present a simple example to illustrate the accuracy of the proposed approximations.

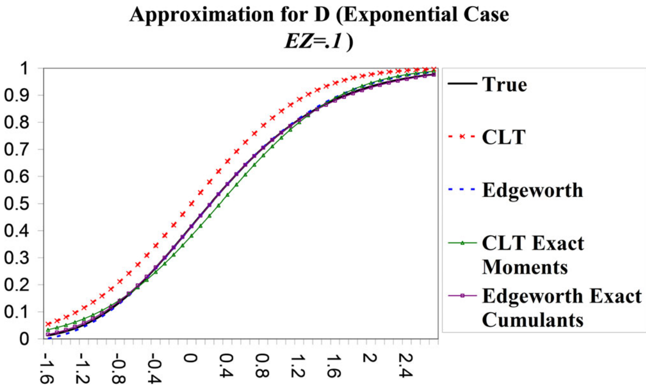

FIGURE 1 Diagram illustrating the fit of the Edgeworth approximation

Example Suppose that $X_1 \sim \lambda Exp(1)$ and $Z_1 \sim \gamma Exp(1)$. Under these assumptions it follows (see, Vervaat (1979), example 3.8.2, and Gjessing and Paulsen (1997)) that

$$D = \sum_{k=0}^{\infty} \exp\left(-\sum_{j=0}^{k-1} Z_j\right) X_k \sim \lambda\Gamma(1/\gamma + 1, 1),$$

where $\Gamma(1/\gamma + 1, 1)$ represents a random variable with distribution gamma with the parameters given. In order to illustrate the numerical fit of our approximation, we consider the case in which $\lambda = 1$ and $\gamma = 0.1$ and $\gamma = 0.5$ respectively. Figure 1 compares the CLT and Edgeworth approximations developed against the true distribution of D (the x-axis is displayed centered and standardized so that the line labeled as "CLT" corresponds to the cumulative distribution function of a standard Gaussian).

We now provide the rigorous statement supporting approximation (31). Our first result provides an asymptotic expansion for $\psi_\alpha(\theta) = \log E \exp\left(i\theta\alpha^{-1/2}(\alpha D(\alpha) - \lambda/\gamma)\right)$ in powers of $\sqrt{\alpha}$.

Lemma 3 *Assume ED1 to ED3 (or ED1′, ED2, and ED3). Then, there exists $\delta > 0$ for which*

$$\psi_\alpha(\theta) = \left(\frac{\mu_Z^{(2)}\lambda}{2\gamma^2} + O(\alpha)\right) i\theta\alpha^{1/2} + \left(\frac{1}{2\gamma\alpha}\left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) + O(1)\right)\frac{(i\theta)^2}{2}\alpha + \left(\frac{c_3}{\alpha} + O(1)\right)\frac{(i\theta)^3}{6}\alpha^{3/2} + o\left(\alpha^{1/2}\right),$$

uniformly in $\theta \in (-\delta, \delta)$ ($\delta > 0$), where

$$3\gamma c_3 = \kappa_X^{(3)} - 2\kappa_{21}\frac{\lambda}{\gamma} + 3\kappa_{12}\frac{\lambda^2}{\gamma^2} + 3\left(\sigma_Z^2\frac{\lambda}{\gamma^2} - \frac{\sigma_{XZ}}{\gamma}\right) \times \left(\sigma_X^2 - 2\frac{\lambda}{\gamma}\sigma_{XZ} + \frac{\lambda^2}{\gamma^2}\sigma_Z^2\right) - \frac{\kappa_Z^{(3)}\lambda^3}{\gamma^3}.$$

Proof The idea is to write

$$\phi_\alpha(\theta) = \exp(i\theta\lambda/\gamma\sqrt{\alpha})\phi(\theta\sqrt{\alpha}, \alpha),$$

where $\phi_\alpha(\theta) \triangleq \exp(\psi_\alpha(\theta))$ and $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$. Notice that $\phi(\theta,\alpha)$ satisfies

$$\phi(\theta,\alpha) = E\left(\exp\left(i\theta\left(X_1 + \exp\left(-\alpha Z_1\right) D_1(\alpha)\right)\right)\right),$$

with $D_1(\alpha)$ independent of (X_1, Z_1). Thus, we have,

$$\begin{aligned}\phi(\theta,\alpha) &= E\left(\exp\left(i\theta\left(X_1 + \exp\left(-\alpha Z_1\right) D_1(\alpha)\right)\right)\right)\\ &= E\left(E\left(\exp\left(i\theta\left(X_1 + \exp\left(-\alpha Z_1\right) D_1(\alpha)\right)\right)\middle| X_1, Z_1\right)\right)\\ &= E\left(E\left(\exp\left(i\theta X_1\right)\phi\left(\theta\exp\left(-\alpha Z_1\right)\alpha\right)\middle| X_1, Z_1\right)\right)\\ &= E\left(\exp\left(i\theta X_1\right)\phi\left(\theta\exp\left(-\alpha Z_1\right), \alpha\right)\right).\end{aligned}$$

Using the Taylor development for characteristic functions (see Feller (1971) sec. XV.5 and Breiman (1992), prop. 8.44) applied to $\phi(\theta,\alpha)$ and $\phi_\alpha(\theta)$, together with the moment conditions implied by assumptions ED1 (or ED1′) to ED3, we arrive at the expression stated for $\psi_\alpha(\theta)$. ∎

Lemma 4 *Under assumptions ED1 to ED3 (or assumptions ED1′, ED2, and ED3), $\phi(\theta,\alpha) \triangleq E\exp(i\theta D(\alpha))$ satisfies*

$$|\phi(\theta,\alpha)| = o\left(\alpha^{1/2}\right)$$

as $\alpha \to 0$ uniformly in θ over compact sets not containing the origin.

Proof Let $\phi_X(\theta, Z_1) = E\left(e^{i\theta X_1}\middle| Z_1\right)$, set $S_k = Z_1 + \ldots + Z_k$, and define $T_\alpha = \inf\{k : S_k > 1/\alpha\}$. Then,

$$\begin{aligned}|\phi(\theta,\alpha)| &= \left|E\left(E\left(\exp\left(i\theta\sum_{k=1}^{\infty} X_k\exp\left(-\alpha S_{k-1}\right)\right)|Z\right)\right)\right|\\ &= \left|E\left(\Pi_{k=1}^{\infty}\phi_X\left(\theta e^{-\alpha S_{k-1}}, Z_k\right)\right)\right|\\ &\le E\left(\Pi_{k=1}^{\infty}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}} Z_k\right)\right|\right)\\ &\le E\left(\Pi_{k=1}^{T_\alpha-1}\left|\phi_X\left(\theta e^{-\alpha S_{k-1}} Z_k\right)\right|\right)\\ &\le E\left(\Pi_{k=1}^{T_\alpha-1}\left|\Delta\left(\theta Z_k\right)\right|\right),\end{aligned}$$

where $\Delta(\theta, Z_1) = \sup\{|\phi_X(\theta^*, Z_1)| : |\theta^*| > |\theta e^{-1}|\}$. Since the distribution of X_1, given Z_1, is non-lattice, we must have that $0 \le \Delta(\theta, Z_1) < 1$. So,

$$\begin{aligned}|\phi(\theta,\alpha)| &\le E\left(\Pi_{k=1}^{T_\alpha-1}\left|\Delta\left(\theta, Z_k\right)\right|\right)\\ &\le P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right)\\ &\quad + E\left(\Pi_{k=1}^{T_\alpha-1}\left|\Delta\left(\theta Z_k\right)\right|; \alpha\left|T_\alpha - 1/\alpha\gamma\right| \le \varepsilon\right)\\ &\le P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right)\\ &\quad + E\left(\left|\Delta\left(\theta, Z_1\right)\right|^{1/\alpha(1/\gamma-\varepsilon)-1}\right).\end{aligned}$$

Since condition ED1 (ED1') implies that $0 < EZ_1 < \infty$ and $Var(Z_1) < \infty$, we have that $\left(\alpha^{1/2}\left|T_\alpha - \frac{1}{\alpha\gamma}\right|\right)^2$ is uniformly integrable (see Gut (1988), p. 92.) In particular, this implies, using Chebyshev's inequality, that

$$P\left(\alpha\left|T_\alpha - \frac{1}{\alpha\gamma}\right| > \varepsilon\right) = O(\alpha).$$

Finally, if we choose $\varepsilon > 0$ small enough so that $c \triangleq 1/\gamma - \varepsilon > 0$, we must show (for θ not in a neighborhood of the origin) that

$$E\left(|\Delta(\theta, Z_1)|^{c/\alpha}\right) = o(\sqrt{\alpha}).$$

Let $W_{\delta'} = -\log\left(\max\left(|\Delta(\theta Z_1)|, \delta'\right)\right)$ for a given $\delta' > 0$ and select $\beta = c/\alpha$. Then,

$$E\left(|\Delta(\theta, Z_1)|^\beta\right) \le E(\exp(-\beta W_{\delta'})) = \int_0^\infty \exp(-u)P(u/\beta > W_{\delta'})\,du.$$

Thus,

$$\beta E\left(|\Delta(\theta, Z_1)|^\beta\right) \le \int_0^\infty \exp(-u)\beta P(u/\beta > W_{\delta'})\,du.$$

Fix $\varepsilon > 0$ and write

$$\begin{aligned}\beta E&\left(|\Delta(\theta, Z_1)|^\beta\right)\\ &\le \int_0^\varepsilon \exp(-u)\beta P(u/\beta > W_{\delta'})\,du\\ &\quad+\int_\varepsilon^\infty u\exp(-u)\beta/uP(u/\beta > W_{\delta'})\,du\\ &\le \beta P(\varepsilon/\beta > W_{\delta'}) + \int_\varepsilon^\infty u\exp(-u)\beta/uP(u/\beta > W_{\delta'})\,du.\end{aligned} \tag{33}$$

We want to apply Fatou's lemma in the form

$$\overline{\lim_{\beta\to\infty}}\int_\varepsilon^\infty u\exp(-u)\beta/uP(u/\beta > W_{\delta'})\,du \le \int_\varepsilon^\infty \overline{\lim_{\beta\to\infty}}\, u\exp(-u)\beta/uP(u/\beta > W_{\delta'})\,du.$$

In order to justify this application of Fatou, we must show that

$$0 \le \beta/uP(u/\beta > W_{\delta'}) \le M$$

for some $M > 0$ for $u \in \varepsilon, \infty]$, and β large. By right continuity and the existence of left limits, it suffices to show that

$$\overline{\lim_{\beta\to\infty}}\frac{P(h > W_{\delta'})}{h} < \infty.$$

But

$$\begin{aligned}\overline{\lim_{h\to 0}}\frac{P(h > W_{\delta'})}{h} &= \overline{\lim_{h\to 0}}\frac{P(h > -\log(\max(|\Delta(\theta Z_1)|, \delta')))}{h}\\ &= \overline{\lim_{h\to 0}}\frac{P(\exp(-h) < |\Delta(\theta, Z_1)|)}{h}\\ &= \overline{\lim_{h\to 0}}\frac{P(|\Delta(\theta, Z_1)| > 1-h)}{h} < \infty,\end{aligned}$$

by virtue of assumption ED3. This is what we require in order to apply Fatou's lemma. Consequently, we have

$$\overline{\lim_{\beta\to\infty}}\,\beta E\left(|\Delta(\theta, Z_1)|^\beta\right) < \infty,$$

which implies

$$\overline{\lim_{\beta\to\infty}}\sqrt{\beta}E\left(|\Delta(\theta, Z_1)|^\beta\right) = 0,$$

and this is what we needed to conclude the proof of the lemma. ■

We now are ready to prove Theorem 5.

Proof of Theorem 5 The proof of this theorem follows closely the steps of Feller (1971), p. 512. To simplify the exposition, let us consider $E(X_1) = 0$ and $E\left(X_1^2\right) = 2\gamma$ and the X_k's independent of the Z_k's. This simplification is helpful in the development of a local expansion. As we shall see from the proof, the adaptation of the present proof to the case in which X_k and Z_k are dependent is straightforward using the corresponding local expansion given in Lemma 3. Once this local expansion is applied, the key part of the proof involves invoking Lemma 4, we will provide explicit guidance for the part of the proof in which the simplifying assumptions are used. Let $\widehat{G}(\theta) = e^{-\theta^2/2}\left(1 + (i\theta)^3\kappa_X^{(3)}\sqrt{\alpha}/(18\gamma)\right)$. Esséen's lemma applies here since

$$G(x) = \Phi(x) - \frac{\kappa_X^{(3)}}{18\gamma}\sqrt{a}\left(x^2 - 1\right)\eta(x)$$

is bounded uniformly by some constant C. Also

$$\widehat{G}(0) = 0 \text{ and } \left.\frac{d\widehat{G}(\theta)}{d\theta}\right|_{\theta=0} = 1.$$

Therefore, defining $F_\alpha(x) = P\left(\alpha^{-1/2}(\alpha D(\alpha) - \lambda/\gamma) \le x\right)$, we have

$$|F_\alpha(x) - G(x)| \le \frac{1}{\pi}\int_{-T}^{T}\frac{1}{|\theta|}|\phi(\sqrt{\alpha}\theta, \alpha) - \widehat{G}(\theta)|d\theta + \frac{24C}{\pi T}.$$

Let $T = M/\sqrt{\alpha}$, for some $M > 0$ big. Then, for any $\delta > 0$ small, we have

$$|F_\alpha(x) - G(x)| \le I_1 + I_2 + I_3 + \sqrt{\alpha}\frac{24C}{\pi M},$$

where

$$I_1 = \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}}\frac{1}{|\theta|}\left|\phi(\sqrt{\alpha}\theta, \alpha) - \widehat{G}(\theta)\right|d\theta,$$

$$I_2 = \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}}\frac{1}{|\theta|}\left|\phi(\sqrt{\alpha}\theta, \alpha) - \widehat{G}(\theta)\right|d\theta,$$

$$I_3 = \frac{1}{\pi}\int_{-M/\sqrt{\alpha}}^{-\delta/\sqrt{\alpha}}\frac{1}{|\theta|}\left|\phi(\sqrt{\alpha}\theta, \alpha) - \widehat{G}(\theta)\right|d\theta.$$

Observe that

$$I_2 \le \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi(\sqrt{\alpha}\theta,\alpha)\right|d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\hat{G}(\theta)\right|d\theta$$
$$= \frac{1}{\pi}\int_{\delta}^{M} \frac{1}{|\theta|}\left|\phi(\theta,\alpha)\right|d\theta + \frac{1}{\pi}\int_{\delta/\sqrt{\alpha}}^{M/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\hat{G}(\theta)\right|d\theta.$$

By virtue of Lemma 4, it is clear that I_2 goes to zero faster than $\sqrt{\alpha}$, similarly for I_3; the argument here would be identical even if we deal with dependent X_k and Z_k. Thus, we just have to study I_1, and this is the part which is easier to explain (notationally) under our simplified assumptions. Let

$$\zeta(\theta,\alpha) \triangleq \log(\phi(\theta,\alpha)) + \frac{\theta^2\gamma}{(1-m(-2\alpha))},$$

where $m(-\upsilon) = E\left(e^{-\upsilon Z_1}\right)$. Hence, we can write

$$I_1 = \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\phi(\sqrt{\alpha}\theta,\alpha) - \hat{G}(\theta)\right|d\theta$$
$$= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}\left|\exp\left(\zeta(\sqrt{\alpha}\theta,\alpha) - \frac{\theta^2\gamma}{(1-m(-2\alpha))}\right) - \hat{G}(\theta)\right|d\theta$$
$$= \frac{1}{\pi}\int_{-\delta/\sqrt{\alpha}}^{\delta/\sqrt{\alpha}} \frac{1}{|\theta|}e^{-\theta^2/2}\left|e^{\left(\zeta(\sqrt{\alpha}\theta,\alpha)-\frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)\right)} - 1 - \frac{(i\theta)^3\kappa_X^{(3)}\sqrt{\alpha}}{18\gamma}\right|d\theta.$$

Using Feller (1971), p. 507, we have that for any $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ complex numbers,

$$\left|e^{\widetilde{\beta}_1} - 1 - \widetilde{\beta}_2\right| \le \left(\left|\widetilde{\beta}_1 - \widetilde{\beta}_2\right| + \frac{1}{2}\widetilde{\beta}_2^2\right)\exp(\upsilon), \qquad (34)$$

where $\upsilon \ge \max\left(\left|\widetilde{\beta}_1\right|, \left|\widetilde{\beta}_2\right|\right)$. Given $\varepsilon > 0$, we can choose $\delta > 0$ small enough so that when $|\theta\sqrt{\alpha}| < \delta$ (as in Feller (1971), p. 507), the following three properties hold. First,

$$\left|\zeta(\theta\sqrt{\alpha},\alpha) - \frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}\right| \le \varepsilon\frac{\theta^3\alpha^{3/2}}{|(1-m(-3\alpha))|}$$
$$\le \varepsilon K_1\theta^3\alpha^{1/2}$$

for α small enough and some constant K_1 independent of α and ε (because $\frac{\alpha^{3/2}\kappa_X^{(3)}}{(1-m(-3\alpha))}$ is the cumulant of order 3 for the random variable $\sqrt{\alpha}D(\alpha)$, which is well defined for $\alpha > 0$ small enough). Second, δ can also be chosen satisfying that if $|\theta\sqrt{\alpha}| < \delta$

$$|\zeta(\theta\sqrt{\alpha},\alpha)| < \frac{1}{2}\frac{\gamma\alpha\theta^2}{(1-m(-2\alpha))} \le \frac{K_2}{3}\theta^2$$

for some $K_2 > 0$ for α small enough. And, third, δ can be chosen also with the property that

$$\left|\frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}\right| < \frac{K_2}{3}\theta^2.$$

Notice that

$$\left|e^{\left(\zeta(\sqrt{\alpha}\theta,\alpha)-\frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)\right)} - 1 - \frac{(i\theta)^3\kappa_X^{(3)}}{18\gamma}\right|$$
$$\le \left|e^{\left(\zeta(\sqrt{\alpha}\theta,\alpha)-\frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))}-1\right)\right)} - 1 - \frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}\right|$$
$$+ \left|\frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))} - \frac{(i\theta)^3\kappa_X^{(3)}}{18\gamma}\sqrt{\alpha}\right|,$$

and observe that

$$\left|\frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))} - \frac{(i\theta)^3\kappa_X^{(3)}}{18\gamma}\sqrt{\alpha}\right| \le \sqrt{\alpha}|\theta|^3 o(1),$$

as $\alpha \to 0$ (uniformly for $|\theta| \le \delta$ with $\delta > 0$ sufficiently small). Finally, we apply inequality (34) with $\widetilde{\beta}_1 = \zeta(\sqrt{\alpha}\theta,\alpha) - \frac{\theta^2}{2}\left(\frac{\alpha\gamma}{(1-m(-2\alpha))} - 1\right)$ and $\widetilde{\beta}_2 = \frac{\alpha^{3/2}(i\theta)^3\kappa_X^{(3)}}{3!(1-m(-3\alpha))}$ for $\delta > 0$ small enough so that

$$I_1 \le \frac{\varepsilon}{\pi}K_1\sqrt{\alpha}\int_{-\infty}^{\infty}\theta^2 e^{-\theta^2/6}d\theta + \frac{\alpha}{\pi}K_1^2\int_{-\infty}^{\infty}e^{-\theta^2/6}\theta^6 d\theta$$
$$+ \frac{\sqrt{\alpha}}{\pi}o(1)\int_{-\infty}^{\infty}|\theta|^3 e^{-\theta^2/6}d\theta.$$

Hence we conclude that

$$\limsup_{\alpha\to 0}\frac{1}{\sqrt{\alpha}}\sup_x|F_\alpha(x) - G(x)| \le \varepsilon K,$$

for some constant K. Since ε was arbitrary, this concludes the proof of the theorem. ∎

4.2 | The continuous-time setting

As noted earlier, we assume that $\Lambda = (\Lambda(t) : t \ge 0)$ is a Levy process and suppose that $Y = (Y(s) : s \ge 0)$ is a time-homogeneous Ito diffusion process living in an open subset $\mathcal{S}$ of $\mathbb{R}^d$ (see, for instance, Stroock and Varadhan (2006) chapters 4 and 8). Moreover, we shall assume that $Y(\cdot)$ possesses a smooth transition semigroup. That is, we assume that for each $y \in \mathcal{S}$, $P_y(Y(s) \in \cdot)$ has a continuously differentiable density for each $s > 0$. This can be guaranteed by means of hypoellipticity; see for instance Nualart (2006) p. 129. The extended generator of the process Y, which is denoted in this section as A_Y, is defined as in Section 2. We also set $\psi_\Lambda(i\theta) = \log E\exp(i\theta\Lambda(1))$.

We further assume appropriate ergodicity and smoothness assumptions. In particular, we assume:

EC1 Λ and Y are independent and the distribution of $\Lambda(1)$ is non-lattice and $E|\Lambda(1)|^p < \infty$ for some $p > 3$.

EC2 There exists $x_0 \in \mathcal{S}$ such that for any $x \in \mathcal{S}$ and any open set B whose closure contains x_0 we have

$$\underline{\lim}_{t\to\infty} P_x(Y(t) \in B) > 0. \qquad (35)$$

EC3 There exists a continuous, unbounded function $w : \mathbb{R}^d \to [1, \infty)$ (in the sense that $\sup_x w(x) = \infty$), a continuous $v : \mathbb{R}^d \to [0, \infty)$ with $\exp(v) \in D(A_Y)$, and constants $\delta, b > 0$ such that for some ball C (with appropriate center and radius) we have

$$\exp(-v)A_Y \exp(v) \leq -\delta w + bI_C.$$

EC4 $\widetilde{\gamma}(\cdot) : \mathbb{R}^d \to (0, \infty)$ is a positive continuous mapping and that

$$\widetilde{\gamma}(x)I(w(x) > r)/w(x) \to 0$$

as $r \nearrow \infty$.

Finally, define

$$\Gamma(t) = \int_0^t \widetilde{\gamma}(Y(s))ds.$$

Assumption EC2 ensures irreducibility and aperiodicity of Y (see proposition 2.2 from Kontoyiannis and Meyn (2017)). Moreover, under assumptions EC2 to EC4, we can guarantee (see Kontoyiannis and Meyn (2005) theorem 2.1) that $E_y\widetilde{\gamma}(Y(t)) = \gamma + O(e^{-rt})$ as $t \to \infty$, for some $r > 0$.

We shall see later that under EC3, there exists a solution pair (u, ψ_Γ) to the generalized eigenvalue problem

$$(A_Y u)(y, z) = \left(\psi_\Gamma(z) - z\widetilde{\gamma}(y)\right) u(y, z), \tag{36}$$

for all z in a neighborhood of the origin in the complex plane. The function $u(\cdot)$ is unique up to constant multiples (we select $u(\cdot)$ so that $u(y, 0) = 1$). Further, under assumption EC3, additional regularity properties hold for the pair (u, ψ_Γ). In particular, we have that $\psi'_\Gamma(0) = \gamma > 0$ and therefore we can define $\chi(\cdot)$ in a neighborhood of the origin via $\chi(z) = -\psi_\Gamma^{-1}(-\psi_\Lambda(z))$, which implies $\chi'(0) = \lambda/\gamma$, with $\lambda = E\Lambda(1)$.

Our goal in this section is to provide rigorous support for the approximation

$$\begin{aligned} P_{y_0}(D \leq y) &\approx P\left(N\left(\lambda/\gamma, \chi^{(2)}(0)/2\right) \leq y\right) \\ &\quad - \sqrt{\gamma}\frac{\lambda}{\gamma}u_z(y_0, 0)\,\eta\left((y - \lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right) \\ &\quad - \frac{\sqrt{\gamma}}{18}\chi^{(3)}(0)H\left((y - \lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right), \end{aligned} \tag{37}$$

when the force of interest γ is small, where $u_z(y_0, 0)$ represents $\partial u(y_0, 0)/\partial z$.

Just as in the discrete time case, we shall make rigorous the approximation (37) in the context of small interest rates for a suitably parameterized family of discounted rewards. In particular, we shall prove the approximation

$$\begin{aligned} &P(\sqrt{\alpha}(D(\alpha) - \lambda/(\gamma\alpha)) \leq y) \\ &\quad = P\left(N\left(0, \chi^{(2)}(0)/2\right) \leq y\right) \\ &\quad - \sqrt{\gamma\alpha}\frac{\lambda}{\gamma}u_z(y_0, 0)\,\eta\left(y\sqrt{2/\chi^{(2)}(0)}\right) \\ &\quad - \frac{\sqrt{\gamma\alpha}}{18}\chi^{(3)}(0)H\left(y\sqrt{2/\chi^{(2)}(0)}\right) + o\left(\alpha^{1/2}\right) \end{aligned}$$

as $\alpha \searrow 0$, where

$$D(\alpha) = \int_0^\infty \exp(-\alpha\Gamma(t))d\Lambda(t).$$

In order to provide the proof of our result, we first collect some regularity properties of Y and the pair (u, ψ_Γ).

Proposition 1 *Assume EC2 to EC4, then exists $\eta > 0$ such that:*

i) *For each y, both $\psi_\Gamma(\cdot)$ and $u(y, \cdot)$ are analytic in the complex neighborhood $N_\eta = \{z \in C : |z| < \eta\}$.*
ii) *For each $0 \leq \theta < \eta$, we have that $\psi_\Gamma(\theta) > 0$ and $u(\cdot, \theta)$ can be taken to be strictly positive.*
iii) *We have $|u(y, z)| \leq \exp(a\eta(v(y) + 1))$ and $|u_z(y, z)/u(y, z)| \leq a\eta(v(y) + 1)$ and $a > 0$.*

In addition:

iv) *For each $\beta \in (0, 1)$ we have*

$$\exp(-\beta v)A_Y \exp(\beta v) \leq -\delta\beta w + b\beta I_C$$

v) *For each y_0 there exists $\delta > 0$ such that.*

$$\sup_{t>0} E_{y_0} \exp(\delta v(Y(t))) < \infty.$$

Proof These properties follow from Kontoyiannis and Meyn (2005). They provide explicit translation of their results from the discrete to the continuous case in some cases but they mostly state their results in discrete time. Nevertheless, since we are assuming that $Y(\cdot)$ is an Ito diffusion with a smooth transition semigroup, the adaptation of their analysis to our setting is easy (although somewhat lengthy). Alternatively, one might invoke the results from Kontoyiannis and Meyn (2017) which approximate spectral quantities of diffusions by Hidden-Markov modes. The approach that we follow here is to take advantage of the properties from Kontoyiannis and Meyn (2005). So, parts i) and ii) follow from theorem 3.1 in Kontoyiannis and Meyn (2005); part iii) is a consequence of proposition 4.4 and equation (2.2). Part iv) follows from part iii) and the analysis of theorem 3.4 and proposition 4.5. Finally, part v) follows as in part ii) of theorem 3.4. ■

Proposition 2 *If assumption EC3 holds then there exists $\varepsilon > 0$ such that for all $\beta \in (0, 1)$,*

$$\begin{aligned} &E_{y_0} \exp\left(\int_0^\infty \beta(\delta w(Y(s)) + \alpha v(Y(s))\widetilde{\gamma}(Y(s))) \exp(-\alpha\Gamma(s))ds\right) \\ &\quad \leq \exp\left(b\beta/(\alpha\varepsilon) + v(y_0)\,\beta\right) \end{aligned}$$

Proof Consider the Markov process $(Y, \Gamma) = ((Y(t), \Gamma(t)) : t \geq 0)$. The coordinate Γ evolves according to the differential equation $\Gamma'(t) = \widetilde{\gamma}(Y(t))$ subject to $\Gamma(0) = \mathfrak{g}$. Using Lemma 1 we have that (Y, Γ) has generator

$$(\widetilde{A}f)(y, \mathfrak{g}) = (Af)(y, \mathfrak{g}) + \mathfrak{g}\partial_{\mathfrak{g}} f(y, \mathfrak{g}).$$

Since C is compact and $\widetilde{\gamma}$ is continuous and positive we have that $\inf_{y\in C} \widetilde{\gamma}(y) \geq \varepsilon > 0$. Consequently, if we consider the function $H(y, \mathfrak{g}) = \exp(\beta v(y) \exp(-\alpha\mathfrak{g}))$ for $\beta \in (0, 1)$ we obtain

$$\begin{aligned} H^{-1}\widetilde{A}H &\leq -\delta\beta \exp(-\alpha\mathfrak{g})w(y) - \alpha\beta v(y)\widetilde{\gamma}(y) \exp(-\alpha\mathfrak{g}) \\ &\quad + b\beta \exp(-\alpha\mathfrak{g})I_C \\ &\leq -\delta\beta \exp(-\alpha\mathfrak{g})w(y) - \alpha\beta v(y)\widetilde{\gamma}(y) \exp(-\alpha\mathfrak{g}) \\ &\quad + b\beta\widetilde{\gamma}(y) \exp(-\alpha\mathfrak{g})/\varepsilon \end{aligned}$$

Therefore, combining Lemma 1 with Fatou's lemma, we obtain

$$\begin{aligned} &\exp\left(v\left(y_0\right) \beta \exp(-\alpha\mathfrak{g})\right) \\ &\geq E_{(y_0,\mathfrak{g})} \exp\left(\int_0^\infty \beta(\delta w(Y(s)) \right. \\ &\quad \left. + \alpha v(Y(s))\widetilde{\gamma}(Y(s))) \exp(-\alpha\Gamma(s))ds\right) \\ &\quad \exp(-b\beta \exp(-\alpha\mathfrak{g})/(\alpha\varepsilon)). \end{aligned}$$

The proposition then follows. ■

We now state the main result of this section.

Theorem 6 *Suppose that EC1 to EC4 hold. Then,*

$$\begin{aligned} &P\left(\sqrt{\alpha}\left(D(\alpha) - \chi'(0)/\alpha\right) \leq y\right) \\ &= P\left(N\left(0, \chi^{(2)}(0)/2\right) \leq y\right) \\ &\quad - \sqrt{\gamma\alpha} u_z\left(y_0, 0\right) \eta\left(y\sqrt{2/\chi^{(2)}(0)}\right) \\ &\quad - \frac{\sqrt{\gamma\alpha}}{18} \chi^{(3)}(0) H\left(y\sqrt{2/\chi^{(2)}(0)}\right) + G_\alpha(y); \end{aligned}$$

where $G_\alpha(y) = o(\sqrt{\alpha})$ as $\alpha \searrow 0$, uniformly over y in compact sets.

The proof of the previous theorem parallels its corresponding discrete time analog described in the previous section. We first obtain a local description of $\psi_\alpha(\theta) = \log E \exp(i\theta\sqrt{\alpha}(D(\alpha) - \lambda/(\gamma\alpha)))$.

Lemma 5 *Under assumptions EC1 to EC4 we have that*

$$\begin{aligned} \psi_\alpha(\theta) &= -\frac{\chi^{(2)}(0)}{2}\theta^2 + \sqrt{\alpha}\left(\frac{\chi^{(3)}(0)}{18}(i\theta)^3 - \frac{\lambda}{\gamma} u_z\left(y_0, 0\right) i\theta\right) \\ &\quad + o(\sqrt{\alpha}) \end{aligned}$$

(uniformly in $\theta \in (-\eta, \eta)$, $\eta > 0$).

Proof Using a variant of Lemma 1, we have that for θ in a neighborhood of the origin, the process

$$\begin{aligned} M_t(i\theta) &= \frac{u\left(Y(t), -\chi\left(i\theta e^{-\alpha\Gamma(t)}\right)\right)}{u\left(y_0, -\chi\left(i\theta e^{-\alpha\Gamma(t)}\right)\right)} \\ &\quad \times \exp\left(\int_0^t \psi_\Lambda\left(i\theta e^{-\alpha\Gamma(s)}\right) ds \right. \\ &\quad \left. - \int_0^t \chi\left(i\theta e^{-\alpha\Gamma(s)}\right) d\Gamma(s)\right) \\ &\quad \times \exp\left(-\alpha\int_0^t i\theta e^{-\alpha\Gamma(s)} \frac{u_\theta\left(Y(s), -\chi\left(i\theta e^{-\alpha\Gamma(s)}\right)\right)}{u\left(Y(s), -\chi\left(i\theta e^{-\alpha\Gamma(s)}\right)\right)} \right. \\ &\quad \left. \dot{\chi}\left(i\theta e^{-\alpha\Gamma(s)}\right) d\Gamma(s)\right) \end{aligned}$$

is a local martingale. Note that

$$\begin{aligned} &E_{y_0}\left(E_{y_0}\left(\exp\left(i\theta\int_0^\infty \exp(-\alpha\Gamma(t))d\Lambda(t)\right)\middle|\Gamma\right)\right) \\ &= E_{y_0}\left(\exp\left(\int_0^\infty \psi_\Lambda(i\theta\exp(-\alpha\Gamma(t)))dt\right)\right). \end{aligned}$$

In addition,

$$\begin{aligned} \left|\int_0^t \chi\left(i\theta e^{-\alpha\Gamma(s)}\right) d\Gamma(s)\right| &\leq \int_0^\infty \left|\chi\left(i\theta e^{-\alpha s}\right)\right| ds \\ &= \int_0^\theta \left|\frac{\chi(iy)}{\alpha y}\right| dy < \infty. \end{aligned}$$

We wish to establish that $M_\cdot(i\theta)$ is a uniformly integrable martingale for all θ in a neighborhood of the origin. In view of the previous pair of estimates (after applying Cauchy-Schwarz inequality) we then must show that the random variables $\lambda(t, i\theta)$ defined via

$$\begin{aligned} \lambda(t, i\theta) &= u\left(Y(t), -\chi\left(i\theta e^{-\alpha\Gamma(t)}\right)\right) \\ &\quad \times \exp\left(\int_0^t \frac{u_\theta\left(Y(s), -\chi\left(i\theta e^{-\alpha\Gamma(s)}\right)\right)}{u\left(Y(s), -\chi\left(i\theta e^{-\alpha\Gamma(s)}\right)\right)} d\chi\left(i\theta e^{-\alpha\Gamma(s)}\right)\right) \end{aligned}$$

are uniformly integrable for all $t > 0$ and $\theta \in (-\eta, \eta)$ provided η is chosen small enough. By virtue of Proposition 1, part iv) and because L_p boundedness implies uniform integrability for some $p > 1$, it suffices to verify that there exists $\delta > 0$ such that

$$\sup_{t>0} E_{y_0} \exp(\delta v(Y(t)) \exp(-\alpha\Gamma(t))) < \infty,$$

$$E_{y_0} \exp\left(\delta\alpha\int_0^\infty v(Y(s))\widetilde{\gamma}(Y(s)) \exp(-\alpha\Gamma(s))ds\right) < \infty.$$

However, this follows from part v) of Propositions 1 and 2. On the other hand, the uniform integrability properties established in the previous display imply the identity

$$u(y, z) = E_y\left\{u(Y(t), z) \exp\left[t\left(\psi_\Lambda(z) - \chi(z)\Gamma(t)\right)\right]\right\}, \qquad (38)$$

which is obtained by considering $\alpha = 0$, and $|z|$ sufficiently small in the definition of $M_t(z)$

and noting that $1 = EM_t(z)$. Now, it follows as in lemma 9.2.2 of Stroock and Varadhan (2006) (thanks to the fact that $Y(\cdot)$ has a continuously differentiable transition density and that $Y(\cdot)$, being an Ito diffusion possesses, has the strong Feller property) that for each bounded measurable function $g(\cdot)$, the mapping $(t, x) \to E_x[g(Y(t))]$ is continuous on $(0, \infty) \times \mathcal{S}$. Consequently, it follows from representation (38) that $u(\cdot)$ is continuous over any set of the form $K \times \{z \in C : |z| \leq \eta\}$ with K compact, assuming that $\eta > 0$ is sufficiently small. We then obtain that $u(\cdot, z) \to 0$ as $z \to 0$ uniformly over compact sets and therefore, because $(Y(t) : t \geq 0)$ is tight and $\chi\left(i\theta e^{-\alpha\Gamma(t)}\right) \to 0$ a.s. as $t \to \infty$, we conclude that

$$M_t(i\theta) \to \frac{1}{u(y_0, i\theta)} \exp\left(\int_0^\infty \psi_\Lambda\left(i\theta e^{-\alpha\Gamma(t)}\right) dt - \int_0^\infty \chi\left(i\theta e^{-\alpha t}\right) dt - \xi(\alpha, i\theta)\right)$$

in probability as $t \to \infty$, where

$$\xi(\alpha, i\theta) = \alpha \int_0^\infty i\theta e^{-\alpha\Gamma(t)} \frac{u_\theta\left(Y(t), -\chi\left(i\theta e^{-\alpha\Gamma(t)}\right)\right)}{u\left(Y(t), -\chi\left(i\theta e^{-\alpha\Gamma(t)}\right)\right)} \dot{\chi}\left(i\theta e^{-\alpha\Gamma(t)}\right) d\Gamma(t).$$

Using the uniform integrability established for $\{\lambda(ti\theta) : t > 0, \theta \in -\eta, \eta]\}$ and sending $t \nearrow \infty$ we obtain that for all $\alpha > 0$ sufficiently small

$$\exp\left(\int_0^\infty \chi\left(i\theta e^{-\alpha t}\right) dt\right) u(y_0, i\theta) = E_{y_0} \exp\left(\int_0^\infty \psi_\Lambda\left(i\theta e^{-\alpha\Gamma(t)}\right) dt - \xi(\alpha, i\theta)\right).$$

Finally, we claim that

$$\begin{aligned}
&\exp\left(\int_0^\infty \left(\chi\left(\sqrt{\alpha} i\theta e^{-\alpha t}\right) - \sqrt{\alpha} i\theta e^{-\alpha t} \lambda/\gamma\right) dt\right) \times u\left(y_0, -\chi(\sqrt{\alpha} i\theta)\right) \\
&= E \exp\left(\int_0^\infty \psi_\Lambda\left(i\theta\sqrt{\alpha} e^{-\alpha\Gamma(t)}\right) dt - i\theta \frac{\lambda}{\gamma\sqrt{\alpha}} - \xi(\alpha, \sqrt{\alpha} i\theta)\right) \\
&= E \exp(i\theta\sqrt{\alpha}(D(\alpha) - \lambda/(\gamma\alpha))) + o(\sqrt{\alpha})
\end{aligned} \tag{39}$$

uniformly in $\theta \in (-\eta, \eta)$ for some $\eta > 0$. Equality (39) follows because

$$\begin{aligned}
&E\xi(\alpha, \sqrt{\alpha} i\theta) \\
&= \sqrt{\alpha} i\theta\alpha E \int_0^\infty e^{-\alpha\Gamma(t)} \frac{u_\theta\left(Y(t), -\chi\left(\sqrt{\alpha} i\theta e^{-\alpha\Gamma(t)}\right)\right)}{u\left(Y(t), -\chi\left(\sqrt{\alpha} i\theta e^{-\alpha\Gamma(t)}\right)\right)} \times \dot{\chi}\left(\sqrt{\alpha} i\theta e^{-\alpha\Gamma(t)}\right) d\Gamma(t) \\
&= \sqrt{\alpha}\theta \frac{\lambda}{\gamma} E\alpha \int_0^\infty e^{-\alpha\Gamma(t)} u_\theta(Y(t), 0) dt + O(\alpha),
\end{aligned}$$

which implies that

$$\alpha E \int_0^\infty e^{-\alpha\Gamma(t)} u_\theta(Y(t), 0) dt = E \int_0^\infty e^{-\alpha\Gamma(t/\alpha)} u_\theta(Y(t/\alpha), 0) dt \to E u_\theta(Y(\infty), 0)/\gamma = 0.$$

The previous estimate, combined with the asymptotic independence of

$$\int_0^\infty e^{-\alpha\Gamma(t/\alpha)} u_\theta(Y(t/\alpha), 0) dt$$

and

$$\int_0^\infty \left(\psi_\Lambda\left(i\theta\sqrt{\alpha} e^{-\alpha\Gamma(t)}\right) - \lambda i\theta\sqrt{\alpha} e^{-\alpha t\gamma}\right) dt,$$

implies (39). The conclusion of the result follows by expanding the left hand side of (39). ■

The proof of Theorem 6 can be completed along the same lines as in the discrete time case after showing that $\phi(\theta, \alpha) \triangleq E \exp(i\theta D(\alpha))$ goes to zero fast enough for $|\theta| \in (w_0, w_1)$ for any $0 < w_0 < w_1 < \infty$ as the next result shows.

Lemma 6 *Suppose that EC1 to EC4 are in force, then* $\phi(\theta, \alpha) \triangleq E \exp(i\theta D(\alpha))$ *satisfies*

$$\sup_{|\theta| \in (\theta_0, \theta_1)} |\phi(\theta, \alpha)| = o(\sqrt{\alpha}),$$

for all $0 < \theta_0 < \theta_1 < \infty$..

Proof Note that

$$\begin{aligned}
|\phi(\theta, \alpha)| &= \left| E \exp\left(\int_0^\infty \psi_\Lambda(i\theta \exp(-\alpha\Gamma(t))) dt\right)\right| \\
&\leq E \left| \exp\left(\int_0^{\Gamma^{-1}(1/\alpha)} \psi_\Lambda(i\theta \exp(-\alpha\Gamma(t))) dt\right)\right|,
\end{aligned}$$

where $\Gamma^{-1}(1/\alpha) = \inf\{t \geq 0 : \Gamma(t) > 1/\alpha\}$. Define

$$\Delta(\theta) = \sup\left\{|\exp(\psi_\Lambda(i\beta))| : |\beta| > \left|\theta e^{-1}\right|\right\}.$$

Since $\Lambda(1)$ is non-lattice, we have that $\Delta(\theta) \in (0, 1)$. But

$$\begin{aligned}
|\phi(\theta, \alpha)| &\leq E\left(\Delta(\theta)^{\Gamma^{-1}(1/\alpha)}\right) \\
&\leq E\left(\Delta(\theta)^{2/(\gamma\alpha)}\right) + P\left(\Gamma^{-1}(1/\alpha) \leq 2/(\gamma\alpha)\right).
\end{aligned}$$

The analysis of Kontoyiannis and Meyn (2005) yields a large deviations principle for $\alpha\Gamma(t/\alpha)$ as $\alpha \searrow 0$ just by simply applying the Gartner-Ellis theorem. Therefore, since $\alpha\Gamma(t/\alpha) \to t\gamma$ as $\alpha \searrow 0$ we conclude that for sufficiently small $\delta > 0$,

$$\begin{aligned}
&\alpha \log P\left(\alpha\Gamma^{-1}(1/\alpha) \leq 1/\gamma - \delta\right) \\
&= \alpha \log P(\alpha\Gamma(1/\gamma - \delta]/\alpha) \geq 1) \to -I(\delta)
\end{aligned}$$

for some constant $I(\delta) > 0$. Therefore, we actually obtain an exponential rate of convergence instead of the rate $o\left(\alpha^{1/2}\right)$, which is more than we need. ■

As an example of the previous ideas we can consider a perpetuity with discount rates driven by model that is often used in finance, namely a Cox-Ingersoll-Ross (CIR) model.

Example 1 Let κ, μ and σ be positive real numbers. The CIR model follows the stochastic differential equation.

$$dY(t) = \kappa(\mu - Y(t))dt + \sigma Y(t)^{1/2}dB(t)$$

subject to $y_0 > 0$. The generator of $Y(\cdot)$ (restricted to twice continuously differentiable functions) takes the form

$$(A_Y h)(y) = \kappa(\mu - y)h'(y) + \sigma^2 y h''(y)/2.$$

Suppose that $\gamma(y) = y$. Then the solution to (36) is given by

$$u(y,\theta) = \exp(\xi(\theta)y), \quad \psi_\Gamma(\theta) = \kappa\mu\xi(\theta)$$

where $\xi(\theta) = \left(\kappa - \left(\kappa^2 - 2\theta\sigma^2\right)^{1/2}\right)/\sigma^2$. Assumptions EC2 and EC3 can be easily verified using the function $u(\cdot)$ directly. Now, assumption EC4 is not satisfied in this setting, but such an assumption was only imposed in our analysis to guarantee the smoothness of $\psi_\Gamma(\cdot)$ and the uniform integrability in the proof of Lemma 5. Fortunately, since we have an explicit expression for $u(\cdot)$ and $\psi_\Gamma(\cdot)$, the regularity properties follow immediately and one can easily adapt the analysis in the proof of Lemma 5 to our current situation if $\alpha > 0$ is sufficiently close to zero.

Let us conclude by pointing out how the methods discussed here can be adapted to situations where the reward rates are driven by more general Markov processes than the Levy case that we discuss here. In particular, following the same ideas as in Lemma 5, a local expansion for $\psi_\alpha(\theta)$ can be obtained for the case in which

$$D(a) = \int_0^\infty \exp\left(-\alpha\int_0^t \widetilde{\gamma}(Y(s))ds\right)\widetilde{\lambda}(Y(s))ds.$$

In this case, the corresponding generalized eigenvalue problem takes the form

$$(Au)(y,\theta) = (\widetilde{\gamma}(y)\chi(\theta) - \theta\widetilde{\lambda}(y))u(y,\theta). \tag{40}$$

and a formal corrected approximation can be written as

$$\begin{aligned} P(D \le y) \approx\ & P\left(N\left(\lambda/\gamma, \chi^{(2)}(0)/2\right) \le y\right) \\ & - \sqrt{\gamma}u_\theta(y_0,0)\,\eta\left((y-\lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right) \\ & - \frac{\sqrt{\gamma}}{18}\chi^{(3)}(0)H\left((y-\lambda/\gamma)\sqrt{2/\chi^{(2)}(0)}\right). \end{aligned}$$

The only step (in addition to the existence of a solution to (40)) required to make the previous approximation rigorous is to show that for all $0 < \theta_0 < \theta_1 < \infty$, $\sup_{|\theta|\in(\theta_0,\theta_1)} |\phi(\theta,\alpha)| = o(\sqrt{\alpha})$ as in Lemma 6. This essentially involves assuming enough structure to ensure strongly non-lattice properties of D. We have chosen Levy processes in our exposition because they are both natural from a modeling viewpoint and provide a convenient framework in which to easily verify, from the model primitives, the non-lattice conditions that yield the described Edgeworth expansions.

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REFERENCES

Asmussen, S. (2001). Ruin probabilities. World Scientific.

Bédard, D., & Dufresne, D. (2001). Pension funding with moving average rates of return. *Scandinavian Actuarial Journal*, *101*, 1–17.

Blackwell, D. (1962). Discrete dynamic programming. *Annals of Mathematical Statistics*, *33*(2), 719–726.

Blanchet, J. (2004) *Limit theorems and approximations with applications to insurance risk and queueing theory*. [Ph.D. dissertation]. Stanford University.

Breiman, L. (1992). Probability. Addison-Wesley.

Bucklew, K., Kurtz, T., & Sethares, W. (1993). Weak convergence and local stability properties of fixed step size recursive algorithms. *IEEE Transactions on Information Theory*, *39*(3), 966–978.

Campbell, J., Lo, A., & Mackinlay, C. (1999). The econometrics of financial markets. Princeton University Press.

Carmona, P., Petit, F., & Yor, M. (2001). *Exponential functionals of Lévy processes*. In O. Barndorff-Nielsen, T. Mikosch, & S. Resnick (Eds.), Lévy processes: Theory and applications (pp. 41–55). Birkhauser.

Diaconis, P., & Freedman, D. (1999). Iterated random functions. *SIAM Review*, *41*, 45–76.

Dufresne, D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal*, *1*, 39–79.

Embrechts, P., & Goldie, C. (1994). *Perpetuities and random equations*. In P. Mandl & M. Huskova (Eds.), Asymptotic statistics. Proceedings of the 5th Prague Symposium (pp. 75–86). Physica-Verlag.

Ethier, S., & Kurtz, T. (1985). Markov processes: Characterization and convergence. Wiley.

Feller, W. (1971). An introduction to probability models and its applications II. Wiley.

Forniari, F., & Mele, A. (1997). Weak convergence and distributional assumptions for a general class of non-linear ARCH models. *Econometric Reviews*, *16*(2), 205–227.

Gerber, H. (1971). The discounted central limit theorem and its berry-Esséen analogue. *Annals of Mathematical Statistics*, *42*(1), 389–392.

Gjessing, H., & Paulsen, J. (1997). Present value distributions with applications to ruin theory and stochastic equations. *Stochastic Processes and Their Applications*, *71*, 123–144.

Goldie, C. (1991). Implicit renewal theory and tails of solutions of random equations. *Annals of Applied Probability*, *1*, 126–166.

Goldie, C., & Grübel, R. (1996). Perpetuities with thin tails. *The Advances in Applied Probability*, *28*, 463–480.

Grandell, J. (1991). Aspects of risk theory. Springer-Verlag.

Gut, A. (1988). Stopped random walks. Springer-Verlag.

Harrison, M. (1977). Ruin problems with compounding assets. *Stochastic Processes and Their Applications*, *5*, 67–79.

Jacod, J., & Shiryaev, A. (2003). Limit theorems for stochastic processes. Springer-Verlag.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Mathematica*, *131*, 207–248.

Kluppelberg, C., & Kostadinova, R. (2007). Integrated insurance risk models with exponential levy investment. *Insurance: Mathematics and Economics*, *42*, 560–577.

Kontoyiannis, I., & Meyn, S. (2005). Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electronic Journal of Probability*, *10*, 61–123.

Kontoyiannis, I., & Meyn, S. (2017). Approximating a diffusion by a hidden Markov model. *Stochastic Processes and Their Applications*, *127*, 2482–2507.

Kurtz, T., & Protter, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Annals of Probability*, *19*, 1035–1070.

Kurtz, T., & Protter, P. (1996). *Weak convergence of stochastic integrals and differential equations*. In D. Talay & L. Tubaro (Eds.), Probabilistic models for nonlinear partial differential equations. Springer-Verlag.

Maulik, K., & Zwart, B. (2006). Tail asymptotics of exponential functionals of Lévy processes. *Stochastic Processes and Their Applications*, *116*, 156–177.

Miao, Y., Xue, T., & Du, T. (2013). The discounted berry-Esséen analogue for autoregressive processes. *Communications in Statistics - Theory and Methods*, *42*, 2684–2693.

Nelson, D. (1990). ARCH models as diffusion approximations. *Journal of Econometrics*, *45*, 7–38.

Nualart, D. (2006). The Malliavin calculus and related topics. Springer-Verlag.

Nyrhinen, H. (2001). Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Processes and Their Applications*, *92*, 265–285.

Paulsen, J. (1993). Risk theory in a stochastic economic environment. *Stochastic Processes and Their Applications*, *46*, 327–361.

Paulsen, J. (1998). Sharp conditions for certain ruin in a risk process with stochastic return on investments. *Stochastic Processes and Their Applications*, *75*, 135–148.

Pollack, M., & Siegmund, D. (1985). A diffusion and its applications to detecting a change in the drift of Brownian motion. *Biometrika*, *72*, 267–280.

Rogers, L., & Williams, D. (1994). Diffusions, Markov processes and martingales. Cambridge University Press.

Shephard, N. (1996). *Statistical aspects of ARCH and stochastic volatility*. In D. R. Cox, D. V. Hinkley, & O. E. Barndorff-Nielsen (Eds.), Likelihood, time series with econometric and other applications. Chapman & Hall.

Stroock, D., & Varadhan, S. (2006). Multidimensional Diffusion Processes. Springer-Verlag.

Veinott, A. F., Jr. (1969). Discrete dynamic programming with sensitive discount optimality criteria. *Annals of Mathematical Statistics*, *40*(5), 1635–1660.

Vervaat, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Advances in Applied Probability*, *11*, 750–783.

Whitt, W. (1972). Stochastic abelian and Tauberian theorems. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, *22*, 25–267.

Whitt, W. (2001). Stochastic-process limits. Springer-Verlag.

Wilkie, A. (1986). A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries*, *39*, 341–363.

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