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Solutions of Poisson's Equation for Stochastically Monotone Markov Chains

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Abstract: Stochastically monotone Markov chains arise in many applied domains, especially in the setting of queues and storage systems. Poisson's equation is a key tool for analyzing additive functionals of such models, such as cumulative sums of waiting times or sums of rewards. In this paper, we show that when the reward function for such a Markov chain is monotone, the solution of Poisson's equation is monotone. This implies that the value function associated with infinite horizon average reward is monotone in the state when the reward is monotone.

Keywords: Markov chains, Poisson's equation, stochastic monotonicity.

1. Introduction

Many stochastic models arising in queueing and storage applications can be formulated as Markov chains that are stochastically monotone, in the sense that if the process is initialized with more "work" in the system, then the system remains more congested. To be more precise, suppose that $X = (X_n : n \geq 0)$ takes values in a state space $S \subseteq \mathbb{R}$. For $x \in S$, let $P_x(\cdot|X_0 = x)$ be the probability on the path-space of X conditional on $X_0 = x$ and let $E_x(\cdot) = E(\cdot|X_0 = x)$ be its associated expectation. We say that X is *stochastically monotone* if for each $y \in S$, $P_x(X_1 > y)$ is non-decreasing as a function of $x \in S$. Many birth-death chains on $\mathbb{Z}_+$ are stochastically monotone, as is the waiting time sequence $W = (W_n : n \geq 0)$ satisfying the recursion

$$W_{n+1} = [W_n + Z_{n+1}]^+$$

for $n \geq 0$, where $Z_1, Z_2, ...,$ are independent and identically distributed (iid) random variables (rv's), $W_0 \in S = \mathbb{R}_+$, and $[y]^+ \stackrel{\Delta}{=} \max(y, 0)$ for $y \in \mathbb{R}$. Stochastically monotone Markov chains are ubiquitous within single station queueing environments; see [15], [9], and [8] for additional discussion and examples.

Consider such a Markov chain X, having a stationary distribution $\pi = (\pi(dx) : x \in S)$. Assume $r : S \to \mathbb{R}$ is a "reward" function which is non-decreasing, so that $r(x) \leq r(y)$ for $x, y \in S$ with $x \leq y$. This covers the great majority of performance measures used within

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the queueing setting. Let $P = (P(x, dy) : x, y \in S)$ be the one-step transition kernel (or matrix, when S is discrete), and let

$$\pi r = \int_S r(y)\pi(dy)$$

be the steady-state expectation of $r(X_n)$. A fundamental object of interest in the analysis of Markov chains is *Poisson's equation*, in which we wish to find the function g for which

$$(P - I)g = -r_c, \tag{1.1}$$

where r_c is the centered reward function given by $r_c(x) = r(x) - \pi r$.

The solution g to (1.1) plays a key role in the analysis of the cumulative reward functional $S_n(r) = \sum_{j=0}^{n-1} r(X_j)$. In particular, the sequence of rv's defined by

$$g(X_n) + \sum_{j=0}^{n-1} r(X_j) - n \cdot \pi r \tag{1.2}$$

is then a martingale (in the presence of appropriate integrability). The martingale structure then implies that

$$E_x S_n(r) = n\pi r + g(x) - E_x g(X_n).$$

When X is aperiodic and g is π-integrable, this leads to the approximation, valid for large time horizons n, given by

$$E_x S_n(r) \approx n\pi r + g(x) - \pi g.$$

The martingale representation (1.2) also allows one to apply martingale theory to establish central limit theorems (CLT's) and laws of the iterated logarithm (LIL's) for $S_n(r)$; see, for example, [10] and [5].

We note also that the value function $v = (v(x) : x \in S)$ for a Markov decision process (MDP) associated with maximizing the long-run average reward $n^{-1}S_n(r)$ satisfies (1.1), where P is the transition kernel (or matrix) associated with the dynamics of X under the optimal policy.

In view of the importance of Poisson's equation, this paper establishes the following fundamental result. In particular, when X is stochastically monotone and r is non-decreasing, the solution g to Poisson's equation must necessarily also be non-decreasing. This monotonicity property implies, for example, that when constructing approximating numerical schemes for computing MDP value functions or value functions for uncontrolled Markov chains, the approximation should ideally be monotone when X is suitably monotone. When value functions are computed via Monte Carlo simulation, the monotonicity can be imposed as a "shape constraint" on the associated estimator, thereby improving estimator performance; see, for example, [2]. It turns out that the monotonicity of g also plays a key role in studying truncation schemes for numerically computing stationary distributions for stochastically monotone chains; see [6].

Our proof uses coupling arguments that may be of separate interest. Section 2 contains statements of our main results, as well as our proofs.

2. The Monotonicity Results

We discuss here our monotonicity results, starting with a simple proof that covers most examples, and then developing theory that covers richer classes of models. Our theory hinges on the following coupling of the Markov chain starting from $x \in S$ with the chain starting from $y \in S$. In particular, let

$$F(x, y) = P_x(X_1 \leq y)$$

for $x, y \in S$, and set

$$F^{-1}(x, u) = \inf\{z : F(x, z) \geq u\}$$

for $u \in [0, 1]$. For $x \in S$, let $X_0(x) = x$ and put

$$X_{n+1}(x) = F^{-1}(X_n(x), U_{n+1})$$

for $n \geq 0$, where $U_1, U_2, ...$ is an iid sequence of rv's uniformly distributed on $[0, 1]$ under the probability P (having associated expectation E). The key observation is that for each $x \in S$,

$$P((X_j(x) : j \geq 0) \in \cdot) = P_x((X_j : j \geq 0) \in \cdot),$$

so that $X(x) = (X_j(x) : j \geq 0)$ has the same distribution as X under P_x. This coupling goes back to [7].

Because X is stochastically monotone, $F^{-1}(\cdot, u)$ is non-decreasing for each $u \in [0, 1]$. Consequently, $F^{-1}(\cdot, U_{n+1})$ is non-decreasing, and it follows by induction that for each $n \geq 0$, $X_n(x)$ is a non-decreasing function of $x \in S$. Hence, if r is a non-decreasing function $r_c(X_n(x))$ is non-decreasing in $x \in S$. Thus, if $r_c(X_n(x))$ is integrable, it is evident that

$$\sum_{j=0}^{n-1} E_x r_c(X_j)$$

is non-decreasing in x. As a result, if

$$\sum_{j=0}^{\infty} |E_x r_c(X_j)| < \infty \tag{2.1}$$

for $x \in S$, then

$$g(x) \triangleq \sum_{j=0}^{\infty} E_x r_c(X_j) \tag{2.2}$$

must be non-decreasing in $x \in S$. But if (2.1) holds, then it is easy to see that g as defined by (2.2) is a solution of Poisson's equation (1.1). We summarize our discussion thus far with our first result.

Proposition 1. Suppose that X is stochastically monotone and that r is non-decreasing. If (2.1) holds for each $x \in S$, then g as defined by (2.2) is a non-decreasing solution of Poisson's equation (1.1).

As noted in [5], there exist Markov chains X for which (1.1) is solvable, and yet the representation (2.2) for the solution fails to be valid. Consequently, we wish to develop alternative proofs that are more general. The remainder of this paper is largely concerned with extending the theory to this setting. We start by noting that [5] show that when r is π-integrable, X is irreducible, and $S \subseteq \mathbb{Z}$, then

$$g_z(x) = E_x \sum_{j=0}^{\tau-1} r_c(X_j) \tag{2.3}$$

always solves (1.1), where $\tau = \inf\{n \geq 1 : X_n = z\}$ is the first return time to any (fixed) state $z \in S$. Furthermore, it is shown there that (1.2) is P_x-integrable for $x \in S$ and that

$$M_n \triangleq g_z(X_n) + \sum_{j=0}^{n-1} r_c(X_j)$$

is a P_x-martingale adapted to $(\mathcal{F}_n : n \geq 0)$, where $\mathcal{F}_n = \sigma(X_j : j \leq n)$.

We now use our coupling to prove the following result.

Theorem 1. Suppose that X is an irreducible positive recurrent stochastically monotone Markov chain taking values in $S = \mathbb{Z}_+$ having stationary distribution π. If $r : S \to \mathbb{R}$ is a π-integrable non-decreasing function, then

$$g_0(x) = E_x \sum_{j=0}^{\tau-1} r_c(X_j) \tag{2.4}$$

(with $\tau = \inf\{n \geq 1 : X_n = 0\}$) is a non-decreasing solution of Poisson's equation (1.1) and $(M_n : n \geq 0)$ is a P_x-martingale for each $x \in S$.

Proof. Fix $z = 0 \leq x < y$. Expressed in terms of our coupling, the martingale structure of $(M_n : n \geq 0)$ implies that

$$M_n(x) \triangleq g_0(X_n(x)) + \sum_{j=0}^{n-1} r_c(X_j(x))$$

and

$$M_n(y) \triangleq g_0(X_n(y)) + \sum_{j=0}^{n-1} r_c(X_j(y))$$

are both martingales adapted to $(\mathcal{G}_n : n \geq 0)$, where $\mathcal{G}_n = \sigma(X_j(x), X_j(y) : 0 \leq j \leq n)$. For $w \in S$, let $\tau(w) = \inf\{n \geq 0 : X_n(w) = 0\}$. We now wish to apply optional sampling at time $\tau(y)$ to $(M_n(x) : n \geq 0)$ and $(M_n(y) : n \geq 0)$. Assuming temporarily that optional sampling can be applied, we find that

$$g_0(x) = Eg_0(X_{\tau(y)}(x)) + E \sum_{j=0}^{\tau(y)-1} r_c(X_j(x)) \tag{2.5}$$

and

$$g_0(y) = Eg_0(X_{\tau(y)}(y)) + E \sum_{j=0}^{\tau(y)-1} r_c(X_j(y)). \tag{2.6}$$

But under our coupling, $X_{\tau(y)}(x) \leq X_{\tau(y)}(y) = 0$. Since $S = \mathbb{Z}_+$, $X_{\tau(y)}(x) \geq 0$, so it follows that $X_{\tau(y)}(x) = 0$. So,

$$g_0(x) - g_0(0) = E \sum_{j=0}^{\tau(y)-1} r_c(X_j(x))$$

and

$$g_0(y) - g_0(0) = E \sum_{j=0}^{\tau(y)-1} r_c(X_j(y)).$$

The monotonicity of r_c implies that

$$\sum_{j=0}^{\tau(y)-1} r_c(X_j(x)) \leq \sum_{j=0}^{\tau(y)-1} r_c(X_j(y)), \tag{2.7}$$

thereby implying that $g_0(x) \leq g_0(y)$.

We now need to establish the validity of optional sampling. Note that $\tau(y) \wedge n \triangleq \min(n, \tau(y))$ is a bounded stopping time, so it follows from the optional sampling theorem (see, for example, Theorem 6.2.2 in [13]) that

$$EM_{\tau(y)\wedge n}(x) = EM_0(x) = g_0(x). \tag{2.8}$$

The validity of (2.5) follows from (2.8), once we prove that $(M_{\tau(y)\wedge n} : n \geq 0)$ is a uniformly integrable sequence. Note that

$$|M_{\tau(y)\wedge n}(x)| \leq |g_0(X_{\tau(y)\wedge n}(x))| + \sum_{j=0}^{\tau(y)-1} |r_c(X_j(x))|. \tag{2.9}$$

Because r_c is non-decreasing, r_c can be expressed as $r_c(x) = \tilde{r}_c(x) + r_c(0)$, where $\tilde{r}_c$ is non-decreasing and non-negative. Hence,

$$\begin{aligned}\sum_{j=0}^{\tau(y)-1} |r_c(X_j(x))| &\leq \sum_{j=0}^{\tau(y)-1} \tilde{r}_c(X_j(x)) + |r_c(0)|\tau(y) \\ &\leq \sum_{j=0}^{\tau(y)-1} \tilde{r}_c(X_j(y)) + |r_c(0)|\tau(y) \\ &= \sum_{j=0}^{\tau(y)-1} |r_c(X_j(y))| + 2|r_c(0)|\tau(y).\end{aligned}$$

Recall that

$$E\sum_{j=0}^{\tau(y)-1} |r_c(X_j(y))| + 2|r_c(0)|\tau(y) = E_y\sum_{j=0}^{\tau-1}|r_c(X_j)| + 2|r_c(0)|E_y\tau.$$

The latter expectations are finite because r_c is π-integrable and X is positive recurrent; see Section 2 of [5] for details. Hence, the latter term on the right-hand side of (2.9) is integrable (and therefore trivially uniformly integrable).

We now argue that the other term appearing on the right-hand side of (2.9), namely $(g_0(X_{\tau(y)\wedge n}) : n \geq 0)$, is also uniformly integrable. Note that $|g_0(X_{\tau(y)\wedge n}(x))| \to |g_0(X_{\tau(y)}(x))|$ a.s. as $n \to \infty$. As argued earlier, $X_{\tau(y)}(x) = 0$, so uniform integrability follows (see Theorem 4.6.3 of [4]) if

$$E|g_0(X_{\tau(y)\wedge n}(x))| \to |g_0(0)| = 0$$

as $n \to \infty$, which is a consequence of establishing that

$$E|g_0(X_n(x))|I(\tau(y) \geq n) \to 0 \tag{2.10}$$

as $n \to \infty$.

Let $\beta_n(x) = \inf\{j > n : X_j(x) = 0\}$, and observe that

$$g_0(X_n(x)) = E[\sum_{j=n}^{\beta_n(x)-1} r_c(X_j(x))|X_n(x)].$$

In view of the fact that $\beta_n(x) \le \beta_n(y)$, the left-hand side of (2.10) can be upper bounded by

$$\begin{aligned} E\sum_{j=n}^{\beta_n(x)-1} |r_c(X_j(x))| I(\tau(y) > n) &\le E\sum_{j=n}^{\beta_n(x)-1} |r_c(X_j(y))| I(\tau(y) > n) \\ &\le E\sum_{j=n}^{\beta_n(y)-1} |r_c(X_j(y))| I(\tau(y) > n) \\ &= E\sum_{j=n}^{\tau(y)-1} |r_c(X_j(y))| I(\tau(y) > n) \\ &= E_y\sum_{j=n}^{\tau-1} |r_c(X_j)| I(\tau > n) \to 0, \end{aligned}$$

proving (2.10). A similar (but easier) argument proves (2.6), thereby proving the theorem.

Theorem 1 provides a general monotonicity theory for solutions to Poisson's equation when the state space is discrete. We finish this section by developing the corresponding theory when the state space S is a continuous state space. In particular, we assume that $S = \mathbb{R}_+$. Our first result concerns the case where X is a positive recurrent Harris chain; see [11] for a complete discussion of this class of Markov chains. Specifically, we will invoke the following assumption

A1. There exist $b, \lambda > 0$, a probability ϕ on $\mathbb{R}_+$, and non-negative functions $v_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ such that

a) $\sup\{E_x v_i(X_1) : 0 \le x \le b\} < \infty,\ i = 1, 2;$
b) $\sup\{|r(x)| : 0 \le x \le b\} < \infty;$
c) $E_x v_1(X_1) \le v_1(x) - 1,\ x > b;$
d) $E_x v_2(X_1) \le v_2(x) - |r(x)|,\ x > b;$
e) $P_x(X_1 \in \cdot) \ge \lambda\phi(\cdot),\ x \in [0, b].$

We now provide a couple of queueing examples, so as to illustrate condition A1.

Example 1. Let $W = (W_n : n \ge 0)$ be the Markov chain on $S = \mathbb{R}_+$ associated with the waiting time sequence for the $G/G/1$ queue with first in/first out (FIFO) queue discipline (See also the Introduction for a discussion of this model.). Then,

$$F(x, y) = P(Z_1 \le y - x)$$

for $x, y \in \mathbb{R}_+$, where Z_1 is the difference of the service time and inter-arrival time associated with the first customer to enter the queue. For any non-negative function $h : \mathbb{R}_+ \to \mathbb{R}_+$, it is easily verified that

$$E_x h(W_1) = Eh([x + Z_1]^+).$$

Assume that Z_1 has a positive continuous density $f_Z(\cdot)$ on $\mathbb{R}$, and that $EZ_1 < 0$ with $EZ_1^2 < \infty$. (The condition $EZ_1 < 0$ is a natural stability condition for the $G/G/1$ queue and asserts that the arrival rate is strictly less than the service rate.) Note that if we set $v_1(x) = 2x/|EZ_1|$, then

$$\begin{aligned} E_x v_1(W_1) - v_1(x) &= \frac{2}{|EZ_1|} E([x + Z_1]^+ - x) \\ &\to -2 \end{aligned}$$

as $x \to \infty$. Furthermore, if we put $v_2(x) = x^2/|EZ_1|$, then

$$\begin{aligned} \frac{(E_x v_2(W_1) - v_2(x))}{x} &= \frac{1}{|EZ_1|x} E(([x + Z_1]^+)^2 - x^2) \\ &\to -2 \end{aligned}$$

as $x \to \infty$. So, there exists $b < \infty$ for which

$$E_x v_1(W_1) \leq v_1(x) - 1$$

and

$$E_x v_1(W_1) \leq v_2(x) - x$$

for $x > b$. Hence, if $r(x) = x$, conditions a)-d) of A1 are clearly satisfied. Finally, for $y \geq 0$,

$$P_x(W_1 \in dy) = P(Z_1 \leq -x)\delta_0(dy) + f_Z(y - x)dy,$$

where $\delta_0(\cdot)$ is a unit point mass distribution at 0. It follows that for $0 \leq x \leq b$,

$$\begin{aligned} P_x(W_1 \in dy) &\geq P(Z_1 \leq -b)\delta_0(dy) + \inf_{0 \leq x \leq b} f_Z(y - x)dy \\ &\triangleq \lambda\phi(dy), \end{aligned}$$

where

$$\begin{aligned} \phi(dy) &= \frac{(P(Z_1 \leq -b)\delta_0(dy) + \inf_{0 \leq x \leq b} f_Z(y - x)dy)}{\lambda}, \\ \lambda &= P(Z_1 \leq -b) + \int_{\mathbb{R}_+} \inf_{0 \leq x \leq b} f_Z(y - x)dy. \end{aligned}$$

This validates condition e) of A1 for Example 1.

Example 2. Consider an infinite server queue in which arrivals occur according to a renewal process with arrival times $(\Lambda_n : n \geq 0)$, and in which the n'th customer to arrive requires processing time V_n. Suppose that X_n is the maximum remaining service time across all customers present in the system at time Λ_n. Then, $X = (X_n : n \geq 0)$ satisfies the recursion

$$X_{n+1} = \max(X_n - \chi_{n+1}, V_{n+1}),$$

where $\chi_{n+1} = \Lambda_{n+1} - \Lambda_n$ for $n \geq 0$.

We assume that $\Lambda_0 = 0$ and $(\chi_n : n \geq 1)$ is an iid sequence independent of $(V_n : n \geq 1)$, with $EV_1^2 + E\chi_1^2 < \infty$ and for which $P(\chi_1 > y)P(V_1 > y) > 0$ for $y \in \mathbb{R}$. Then, $X = (X_n : n \geq 0)$ is a Markov chain taking values in $S = \mathbb{R}_+$. If, $v_1(x) = 2x/E\chi_1$, note that

$$\begin{aligned} E_x v_1(X_1) - v_1(x) &= \frac{2}{E\chi_1} E \max(-\chi_1, V_1 - x) \\ &\to -2 \end{aligned}$$

as $x \to \infty$. Similarly, if $v_2(x) = x^2/E\chi_1$, then

$$\begin{aligned} \frac{1}{x}\left(Ev_2(X_1) - v_2(x)\right) &= \frac{1}{xE\chi_1} E \max\left((x - \chi_1)^2 - x^2, V_1^2 - x^2\right) \\ &\to -2 \end{aligned}$$

as $x \to \infty$. As in Example 1, there exists $b < \infty$ such that

$$E_x v_1(X_1) \leq v_1(x) - 1$$

and

$$E_x v_2(X_1) \leq v_2(x) - x$$

for $x > b$. Hence, conditions a)-d) are easily seen to hold for Example 2 with $r(x) = x$.

As for condition e), note that

$$\begin{aligned} P_x(X_1 \in dy) &\geq P(V_1 \in dy, x - \chi_1 < y) \\ &= P(V_1 \in dy)P(\chi_1 > x - y) \end{aligned}$$

for $x, y \in \mathbb{R}_+$. Hence, for $0 \leq x \leq b$ and $y > b$,

$$\begin{aligned} P_x(X_1 \in dy) &\geq P(V_1 \in dy)P(\chi_1 \geq 0) \\ &= \lambda\phi(dy), \end{aligned}$$

where

$$\phi(dy) = \frac{P(V_1 \in dy)}{P(V_1 > b)}, y > b$$

and

$$\lambda = P(V_1 > b),$$

establishing that e) also holds for Example 2 with $r(x) = x$.

[1] and [12] noted that condition e) allows one to "split" the transition kernel P over $[0, b]$, so that one can then write

$$P_x(X_1 \in \cdot) = \lambda\phi(\cdot) + (1 - \lambda)Q(x, \cdot) \tag{2.11}$$

for $x \in [0, b]$, where $Q(x, \cdot)$ is a probability on S for each $x \in [0, b]$. The mixture (2.11) then allows one to interpret transitions from $x \in [0, b]$ in terms of a "randomization". In particular, if $X_n \in [0, b]$, then X_{n+1} is distributed according to ϕ with probability λ and distributed according to $Q(X_n, \cdot)$ with probability $1 - \lambda$. Every time that X distributes itself according to ϕ, the Markov chain "regenerates." More precisely, the regeneration time τ is a randomized stopping time, in which τ is adapted to a filtration involving the history of the sequence $((X_j, \beta_j) : j \geq 0)$, where β_j is a Bernoulli rv in which $\beta_j = 1$ whenever the mixture of component ϕ is chosen (with probability λ) and $\beta_j = 0$ otherwise; see p.98-100 of [11] for details.

Let τ be the first time that X distributes itself according to ϕ. We are now ready to state a theorem that provides a continuous state space analog to the discrete state space representation (2.3) for the solution of Poisson's equation.

Theorem 2. Let $X = (X_n : n \geq 0)$ be a Markov chain satisfying A1. Then, X possesses a unique stationary distribution π and r is π-integrable. If $r_c(x) = r(x) - \pi r$ for $x \geq 0$, then

$$E_x \sum_{j=0}^{\tau-1} |r_c(X_j)| < \infty$$

for each $x \in \mathbb{R}_+$ and

$$\tilde{g}(x) = E_x \sum_{j=0}^{\tau-1} r_c(X_j)$$

is a finite-valued solution of Poisson's equation for which

$$\tilde{g}(X_n) + \sum_{j=0}^{n-1} r_c(X_j)$$

is a P_x-martingale for each $x \in \mathbb{R}_+$.

Proof. Condition e) implies that $[0, b]$ is a small set, in the terminology of [11]. In view of conditions a) and c), we may apply Theorem 11.3.4 of [11], thereby ensuring that X has a unique stationary distribution π. Furthermore, conditions a), b), and d), together with Theorem 14.3.7 of [11] imply that

$$\int_{\mathbb{R}_+} \pi(dx)|r(x)| \leq \sup\{E_x v_2(X_1) - v_2(x) + |r(x)| : 0 \leq x \leq b\} < \infty.$$

Since πr is finite-valued, we may put $r_c(x) = r(x) - \pi r$.

In view of conditions a)-d), we may apply the Comparison Theorem (p. 344, [11]) to conclude that

$$E_x \sum_{j=0}^{\tau-1} |r_c(X_j)| \leq v_2(x) + |\pi r| v_1(x) + \beta E_x \sum_{j=0}^{\tau-1} I(X_j \leq b) \tag{2.12}$$

where

$$\beta = \sup\{|\pi r| v_1(x) + v_2(x) : 0 \leq x \leq b\}.$$

With the aid of the splitting idea discussed earlier, we see that

$$P_x(\tau > T_{k+1} | X_0, ..., X_{T_k}) = (1 - \lambda),$$

where T_k is the time step at which X visits $[0, b]$ for the k'th time, and hence

$$P(\tau > T_k) = (1 - \lambda)^k.$$

It follows that

$$E_x \sum_{j=0}^{\tau-1} I(X_j \leq b) = E_x \sum_{j=1}^{\infty} I(\tau > T_j) = \frac{1}{\lambda}.$$

Consequently, (2.12) implies that

$$E_x \sum_{j=0}^{\tau-1} |r_c(X_j)| < \infty \tag{2.13}$$

for all $x \in \mathbb{R}$.

With the knowledge that $\tilde{g}$ is finite-valued, then by conditioning on X_1, we find that

$$\tilde{g}(x) = r_c(x) + \int_{\mathbb{R}_+} \tilde{g}(y) P_x(X_1 \in dy) \tag{2.14}$$

for $x > b$, whereas

$$\tilde{g}(x) = r_c(x) + (1 - \lambda) \int_{\mathbb{R}_+} \tilde{g}(y) Q(x, dy) \tag{2.15}$$

for $x \leq b$.

Since X regenerates at time τ and has distribution ϕ at that time, it is well known (see, for example, [14]) that

$$0 = \pi r_c = \frac{E_\phi \sum_{j=0}^{\tau-1} r_c(X_j)}{E_\phi \tau} = \frac{\int_{\mathbb{R}_+} \phi(dx) \tilde{g}(x)}{E_\phi \tau}, \tag{2.16}$$

(where $E_\phi(\cdot)$ is the expectation under which X_0 has distribution ϕ) so that (2.15) becomes

$$\begin{aligned}\tilde{g}(x) &= r_c(x) + (1-\lambda)\int_{\mathbb{R}_+} \tilde{g}(y)Q(x,dy) + \lambda\int_{\mathbb{R}_+} \tilde{g}(y)\phi(dy) \\ &= r_c(x) + \int_{\mathbb{R}_+} \tilde{g}(y)P_x(X_1 \in dy),\end{aligned}$$

proving that Poisson's equation is solved by $\tilde{g}$.

To prove the martingale property, we note that

$$E_x|r_c(X_n)| = E_x|r_c(X_n)|I(\tau > n) + \sum_{j=1}^{n} P_x(\tau = j)E_\phi|r_c(X_{n-j})| \tag{2.17}$$

and

$$E_\phi|r_c(X_n)| = E_\phi|r_c(X_n)|I(\tau > n) + \sum_{j=1}^{n} P_\phi(\tau = j)E_\phi|r_c(X_{n-j})| \tag{2.18}$$

(where $P_\phi(\cdot)$ is the probability under which X_0 has distribution ϕ). But the π-integrability of r ensures that

$$E_\phi \sum_{j=0}^{\tau-1} |r_c(X_j)| < \infty \tag{2.19}$$

(using the same regenerative identity as in (2.16)), so that $E_\phi|r_c(X_n)|I(\tau > n) < \infty$ for $n \geq 0$. Also, (2.13) guarantees that $E_x|r_c(X_n)|I(\tau > n) < \infty$ for $n \geq 0$. It follows from an induction based on the recursions (2.17) and (2.18) that $r_c(X_n)$ is P_x-integrable for $n \geq 0$.

We can similarly analyze the P_x-integrability of $\tilde{g}(X_n)$. The same inductive argument, based on equations analogous to (2.17) and (2.18), establishes the integrability provided that we show

$$E_x|\tilde{g}(X_n)|I(\tau > n) < \infty$$

and

$$E_\phi|\tilde{g}(X_n)|I(\tau > n) < \infty$$

for $n \geq 0$. Set $\beta_n = \inf\{j > n : X \text{ regenerates at time } j\}$. Then,

$$\begin{aligned}E_x|\tilde{g}(X_n)|I(\tau > n) &= E_x\Big|\sum_{j=n}^{\beta_n - 1} r_c(X_j)\Big|I(\tau > n) \\ &\leq E_x \sum_{j=n}^{\beta_n - 1} |r_c(X_j)|I(\tau > n) \\ &= E_x \sum_{j=n}^{\tau-1} |r_c(X_j)|I(\tau > n) < \infty,\end{aligned}$$

due to (2.13). Similarly,

$$E_\phi|\tilde{g}(X_n)|I(\tau > n) \le E_\phi \sum_{j=n}^{\tau-1} |r_c(X_j)|I(\tau > n) < \infty,$$

due to (2.19). Hence,

$$\tilde{g}(X_n) + \sum_{j=0}^{n-1} r_c(X_j)$$

is P_x-integrable for $n \ge 0$. The martingale property then follows easily from the fact that $\tilde{g}$ solves Poisson's equation.

We now proceed to prove that $\tilde{g}$ is monotone when X and r are suitably monotone.

Theorem 3. Suppose that X is a stochastically monotone Markov chain satisfying A1. If the function r appearing in A1 is non-decreasing, then $\tilde{g}$ is non-decreasing.

Proof. To prove this result, we modify the coupling discussed earlier. In particular, for $0 \le x \le y$, we modify the dynamics of $((X_n(x), X_n(y)) : n \ge 0)$ when $X_n(y) \le b$.

For $w \in \mathbb{R}_+$ and $v \in [0, b]$, put

$$G(v, w) = \frac{F(v, w) - \lambda\phi([0, w])}{1 - \lambda},$$

and note that $1 - G(\cdot, w)$ is non-decreasing for each $w \ge 0$ (since this is also true of $1 - F(\cdot, w)$).

When $X_n(x) \le X_n(y) \le b$, we distribute $X_{n+1}(y)$ according to ϕ with probability λ and put $X_{n+1}(x) = X_{n+1}(y)$. Otherwise, with probability $1 - \lambda$, put $X_{n+1}(x) = G^{-1}(X_n(x), U_{n+1})$ and $X_{n+1}(y) = G^{-1}(X_n(y), U_{n+1})$. This modified coupling preserves the distribution of X under P_x and the distribution of (X, τ) under P_y while maintaining the ordering $X_n(x) \le X_n(y)$ for $n \ge 0$ and forcing $X_\tau(x)$ to equal $X_\tau(y)$.

Because $X(x)$ may visit $[0, b]$ earlier than $X(y)$, it may regenerate and distribute itself according to ϕ at a time τ' earlier than τ. In particular, when $X_n(x) \le b < X_n(y)$ (so that $X(x)$ has the potential to regenerate at time $n + 1$, but $X(y)$ does not), we put $X_{n+1}(x) = F^{-1}(X_n(x), U_{n+1})$ and $X_{n+1}(y) = F^{-1}(X_n(y), U_{n+1})$. A regeneration for $X(x)$ occurs at time $n + 1$ with probability $w(X_n(x), X_{n+1}(x))$, where

$$w(x, y) = \lambda \left[\frac{d\phi}{dP_x(X_1 \in \cdot)}\right](y)$$

is the Radon-Nikodym derivative of $\lambda\phi$ with respect to $P_x(X_1 \in \cdot)$ (which exists because ϕ must be absolutely continuous with respect to $P_x(X_1 \in \cdot)$). Then, the distribution of (X, τ) under P_x matches the distribution of $(X(x), \tau')$.

The rest of the argument follows the proof used to verify Theorem 1. In particular, we consider the martingales

$$\tilde{g}(X_n(x)) + \sum_{j=0}^{n-1} r_c(X_j(x)) \tag{2.20}$$

and

$$\tilde{g}(X_n(y)) + \sum_{j=0}^{n-1} r_c(X_j(y)), \tag{2.21}$$

analogously to the martingales $(M_n(x) : n \geq 0)$ and $(M_n(y) : n \geq 0)$ used in Theorem 1. The key is then to prove that optional sampling can be applied to (2.20) and (2.21) at time τ. The associated uniform integrability argument is essentially identical to that used in Theorem 1, and is therefore omitted, proving the result.

In view of Examples 1 and 2, Theorem 3 implies that Poisson's equation has a non-decreasing solution for $r(x) = x$ for both systems.

In some applications, it is important to know that the solution $\tilde{g}$ to Poisson's equation is continuous on $\mathbb{R}_+$.

Proposition 2. Suppose that X_n is a stochastically monotone Markov chain and that $F^{-1}(\cdot, y)$ is continuous on $\mathbb{R}_+$ for each $y \geq 0$. If the function r appearing in A1 is continuous and non-decreasing, then $\tilde{g}$ is continuous.

Proof. Suppose that $x_n \to x \geq 0$ as $n \to \infty$ with $x_n \leq y$ for all $n \geq 1$. The proof of Theorem 3 establishes that if $x < y$, then

$$\tilde{g}(x_n) - \tilde{g}(x) = E \sum_{j=0}^{\tau-1} [r_c(X_j(x_n)) - r_c(X_j(x))].$$

Since $r_c(F^{-1}(\cdot, U_n))$ is continuous, it follows that

$$\sum_{j=0}^{\tau-1} [r_c(X_j(x_n)) - r_c(X_j(x))] \to 0 \qquad \text{a.s.}$$

as $n \to \infty$. Furthermore, since $r_c(X_j(\cdot))$ is non-decreasing,

$$\sum_{j=0}^{\tau-1} |r_c(X_j(x_n)) - r_c(X_j(x))| \leq 2 \sum_{j=0}^{\tau-1} |r_c(X_j(y))| + 2\tau |r_c(0)|. \tag{2.22}$$

The argument of Theorem 1 proves that (2.22) is integrable, so that the Dominated Convergence Theorem applies. So $\tilde{g}(x_n) \to \tilde{g}(x)$ as $n \to \infty$, proving the continuity of $\tilde{g}$.

Our final result concerns a class of Markov chains on $\mathbb{R}_+$ that need not be Harris recurrent and need not satisfy A1. In particular, we assume that for $0 \leq x \leq y$, there exists $\rho < 1$ such that

$$E|F^{-1}(x, U_1) - F^{-1}(y, U_1)|^2 \leq \rho|x - y|^2; \tag{2.23}$$

such a Markov chain is said to be *contractive on average*. Let $\kappa_n : \mathbb{R}_+ \to \mathbb{R}_+$ be the random mapping defined by

$$\kappa_n(x) = F^{-1}(x, U_n)$$

for $n \geq 1$. Observe that

$$X_n(x) = (\kappa_n \circ \kappa_{n-1} \circ ... \circ \kappa_1)(x)$$

has precisely the same distribution as

$$\tilde{X}_n(x) = (\kappa_1 \circ \kappa_2 \circ ... \circ \kappa_n)(x)$$

for $n \geq 1$. If $X_0(x) = \tilde{X}_0(x), \tilde{X}_n(\cdot)$ enjoys the same monotonicity property as does $X_n(\cdot)$, so that $\tilde{X}_n(x) \leq \tilde{X}_n(y)$.

Assume that r is Lipschitz, so that there exists $c < \infty$ for which

$$|r(x) - r(y)| \leq c^{1/2}|x - y|.$$

Then, (2.23) implies that

$$\begin{aligned}
E|r(\tilde{X}_{k+1}(x)) - r(\tilde{X}_k(x))|^2 &\leq cE|\tilde{X}_{k+1}(x) - \tilde{X}_k(x)|^2 \\
&= cE|F^{-1}((\kappa_2 \circ ... \circ \kappa_{k+1}(x), U_1)) - F^{-1}((\kappa_2 \circ ... \circ \kappa_k)(x), U_1)|^2 \\
&\leq c\rho E|(\kappa_2 \circ ...\kappa_{k+1}(x)) - (\kappa_2 \circ ...\kappa_k)(x)|^2 \\
&= c\rho E|F^{-1}((\kappa_3 \circ ... \circ \kappa_{k+1})(x), U_2) - F^{-1}((\kappa_3 \circ ... \circ \kappa_k)(x), U_2)|^2 \\
&\leq c\rho^2 E|(\kappa_3 \circ ... \circ \kappa_{k+1})(x) - (\kappa_3 \circ ... \circ \kappa_k)(x)|^2 \\
&\leq ... \\
&\leq c\rho^k E|\kappa_{k+1}(x) - x|^2 \\
&= c\rho^k E|\kappa_1(x) - x|^2.
\end{aligned}$$

Hence, if

$$E_x|X_1 - x|^2 < \infty. \tag{2.24}$$

$(r(\tilde{X}_k(x)) : k \geq 0)$ is evidently a Cauchy sequence in the space of square integrable rv's and hence converges in mean square to a limit $r(\tilde{X}_\infty(x))$. [3] show that (2.23) and (2.24) imply that X has a unique stationary distribution π and that $r(\tilde{X}_\infty(x))$ has the distribution of $r(X_0)$ under π. Hence, Cauchy-Schwarz implies that

$$\begin{aligned}
|E_x r_c(X_n)| &= |Er(\tilde{X}_n(x)) - Er(\tilde{X}_\infty(x))| \\
&\le \sum_{j=n}^{\infty} |Er(\tilde{X}_j(x)) - Er(\tilde{X}_{j+1}(x))| \\
&\le \sum_{j=n}^{\infty} E^{1/2}|r(\tilde{X}_j(x)) - r(\tilde{X}_{j+1}(x))|^2 \\
&\le c^{1/2} \sum_{j=n}^{\infty} \rho^{j/2} E^{1/2}|X_1(x) - x|^2,
\end{aligned}$$

and consequently,

$$\sum_{k=0}^{\infty} |E_x r_c(X_k)| < \infty,$$

and a solution g to Poisson's equation can then be defined by (2.2). Since

$$\sum_{j=0}^{\infty} r_c(\tilde{X}_j(x)) \le \sum_{j=0}^{\infty} r_c(\tilde{X}_j(y))$$

for $x \le y$, it follows that g is non-decreasing. Similarly, we find that

$$\begin{aligned}
|E_x r_c(X_n) - E_y r_c(X_n)| &\le c^{1/2} E^{1/2} |X_n(x) - X_n(y)|^2 \\
&\le c^{1/2} \rho^{n/2} |x - y|
\end{aligned}$$

so that

$$|\tilde{g}(x) - \tilde{g}(y)| \le \frac{c^{1/2}}{1 - \rho^{1/2}} |x - y|$$

and hence $\tilde{g}$ is Lipschitz. We have proved our final result.

Theorem 4. Suppose that X is a stochastically monotone Markov chain satisfying (2.23) and (2.24). If r is a non-decreasing Lipschitz function, then g as defined by (2.2) is a finite-valued non-decreasing Lipschitz solution to Poisson's equation (1.1).

3. Conclusion

In this paper, we have developed theoretical results that establish monotonicity of the solution to Poisson's equation when the underlying Markov chain is stochastically monotone and the reward function is monotone, in the setting that the state space of the chain is a subset of the totally ordered real line. Such real-valued Markov chains arise in many queueing-related applications. An interesting generalization of this work would involve development of corresponding theory when the underlying Markov chain is stochastically monotone with respect to a partially ordered state space (as often occurs when a queueing network's state space is vector-valued).

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