On Matching and Thickness in Dynamic Markets

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Abstract

We study dynamic matching in an infinite-horizon stochastic networked market, in which some agents are a priori more difficult to match than others. Agents have compatibility-based preferences and can match either bilaterally, or indirectly through chains. We study how these matching technologies, as well as matching policies, affect efficiency in such markets with different thickness levels.

First, we study myopic matching policies and identify a strong connection between market thickness and the efficiency driven by the matching technology. We show that when “hard-to-match” agents join the market more frequently than “easy-to-match” ones, moving from bilateral matchings to chains significantly increases efficiency. Otherwise, the difference between matching bilaterally or through a chain is negligible.

Second, we show that the lack of thickness cannot be compensated by better matching policies implying that the only way to thicken the market fruitfully is by attracting more agents.

1 Introduction

Many matching markets are naturally dynamic, were agents arrive and match over time. Every year thousands of incompatible patient-donor pairs register to kidney exchange clearinghouses that search periodically for matches between these pairs. Online platforms (dating, online workplace, etc.), labor markets, and even housing markets can be viewed as dynamic

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matching markets. The theory of mechanisms for matching markets has focused mostly on static environments and little is known about clearinghouses for dynamic matching markets.

The matching policy, which determines when to match and who to match, plays an important role in the efficiency of the marketplace. For instance a greedy or myopic policy, which attempts to match agents upon arrival may have short run benefits but may harm agents who are yet to arrive to the market. In particular, a centralized clearinghouse may wait to thicken the market before matches take place. Kidney exchange clearinghouses typically search for matches very frequently: matching is done on a daily basis at the Alliance for Paired Donation and the National Kidney Registry, weekly in the United Network for Organ Sharing and monthly in the Netherlands national program.

In various marketplaces, the matching technology is also instrumental for efficiency. For example, while kidney exchanges were first conducted in 2-way cycles (bilateral matches), the majority of transplants are now conducted through chains initiated by an altruistic donor (Anderson et al. (2015)). Transactions in housing markets may also be viewed as conducted through chains. In some matching markets (such as dating) only bilateral matches take place. This paper is concerned with the effect matching technologies, as well as matching policies, have on the efficiency of dynamic matching markets with different thickness levels.

We address these questions by studying a stylized infinite horizon model with two types of agents distinguished by their difficulty to match. Every period a single agent arrives to the market whose type is either hard-to-match ($H$), or easy-to-match ($E$), with probability $\theta$ and $1 - \theta$, respectively. Easy and hard-to-match agents can be matched by any other agent independently with probability $p_E$ and $p_H$, respectively, where $p_H \ll p_E$. So the fraction of hard-to-match agents joining the market, $\theta$, can be viewed as a measure for the market thickness, with a higher fraction interpreted as a thinner market.

Agents in our model prefer to match as early as possible and are indifferent between acceptable matches. We adopt the average waiting time of agents in steady-state as a measure for efficiency. Agents leave the market only after they match (but see Section 4 for a discussion about a model with departures). Two types of matchings are considered, bilateral and chains, and we assume that only one of these take place. In a bilateral exchange a pair of agents match (with) each other and in a chain an agent is matched by one agent but matches another. We are interested in the behavior of the average waiting time for small values of $p_H$.

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1How to attract agents to the marketplace is a complementary issue (see Ashlagi and Roth (2013) who study how to incentivize hospitals to fully participate in kidney exchange).

2A chain consists of a sequence of patient-donor pairs with the donor of each pair donating to the patient of the next pair. The chain is initiated by an altruistic donor.

3Typically an agent will sell a house to some agent and buy from another.
Recent papers study matching thin (sparse) markets with compatibility based preferences. Anderson et al. (2013) and Akbarpour et al. (2014) analyze optimal policies in infinite-horizon matching markets, however, these papers restrict attention to homogenous markets with ex ante symmetric agents. Markets, however, typically have more heterogeneity. In kidney exchange for example Ashlagi et al. (2013) document that most patients’ sensitization level is either very low or very high. In this paper, we investigate how the thickness/sparsity levels of the marketplace, captured by heterogeneity, affects the choice of policies and types of matchings.

Our main contribution is identifying a tight connection between the fraction of hard-to-match agents joining the market and the effect the matching technology has on efficiency. We show that if easy-to-match agents join the market more frequently than hard-to-match ones ($\theta < 0.5$), myopic matching policies that use a chain, or just bilateral matches, are both approximately optimal. Otherwise, any bilateral matching policy is highly sub-optimal, but a myopic policy that matches through chains (even a single one) is approximately optimal.

More formally, we first analyze a myopic policy which conducts bilateral matches. Because easy-to-match agents in our model can be matched almost instantaneously (relatively to hard-to-match agents), we focus on the average waiting time of hard-to-match agents. For a fixed $p_E$, and sufficiently small values of $p_H$, the average waiting time of hard-to-match agents is of order $\Theta(\frac{1}{p_E p_H})$ for any $\theta < 0.5$, and $\Theta(\frac{1}{p_H^2})$ for any $\theta \geq 0.5$.

Intuitively, the waiting time for agents is inversely proportional to the probability for a bilateral match to occur. When hard-to-match agents are on the long side ($\theta \geq 0.5$), many hard-to-match agents must match with each other, but otherwise almost all hard-to-match agents must match with each other, but otherwise almost all hard-to-match agents match with easy-to-match ones.

We next analyze a myopic policy that conducts matches by constructing a single chain myopically. Whenever an agent is matched, the next agent in the chain is instantaneously selected arbitrarily among all feasible agents, and so forth. Once the chain cannot be continued, the last agent in the chain waits for the next arriving agent that it can match. Note that at every period we end up with a maximal chain, but not necessarily the longest possible one. To simplify the analysis we assume that easy-to-match agents can be matched by any other agent. For any $\theta < 1$, hard-to-match agents will wait on average $\Theta(\frac{1}{p_H})$ periods, and interestingly, when $\theta = 1$, their average waiting time is $\Theta(\frac{\log(1/p_H)}{p_H})$. Again, hard-to-match

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4We note that the model by Anderson et al. (2013) is the special case of our model where $\theta = 1$.

5The more sensitized a patient the harder it is to match.

6We write $f = \Theta(g)$ if there exist constants $a$ and $b$ such that $a g(n) \leq f(n) \leq b g(n)$ for every sufficiently large $n$.

7The myopic policy that we analyze assigns priority to $H$ agents in the presence of ties. However, as will be shown, the average waiting time of $H$ agents remains the same order of magnitude even when ties are broken in favor of $E$ agents.
agents’ average waiting is inversely proportional to their chance to be matched by a random agent.

We further address whether non-myopic matching policies can improve hard-to-match agents’ waiting times. For example, by making easy-to-match agents wait to thicken the market, or by searching for the longest chain (rather than just advancing it myopically) one may potentially increase overall efficiency. We show, however, that no matching policy can reduce the average waiting time of hard-to-match agents without significantly harming easy-to-match agents and thus any attempt to thicken the pool is artificial. We further show that myopic matching policies are approximately optimal. Putting together, we find that when hard-to-match agents arrive more frequently than easy-to-match ones, matching through a chain results in much lower waiting times than matching only bilaterally. Otherwise, bilateral matchings are comparable to matching through chains.

Finally, in Section 4 we briefly discuss several extensions to our model. In Section 4.1 we relax the assumption that arrival rates of easy-to-match and hard-to-match agents are perfectly correlated, and numerical simulations for arrival rates are independent across types. In Section 4.2 we allow agents to depart without being matched and discuss the relation between minimizing the number of unmatched agents and minimizing the average waiting time in the market. In Section 4.3 we further extend our discussion to two-sided markets with asymmetric arrival rates between both sides. In Section 4.4 we apply our findings to decentralized markets.

1.1 Related work

The first strand of research, related directly to our work, is the growing area of dynamic matching markets in networks, where agents are matched to other agents. Several papers in this strand focused on compatibility-based preferences. Ünver (2010) analyzes a dynamic kidney exchange model in which preferences are based on compatibilities determined only by blood types. He finds that a myopic mechanism that uses only 2 and 3-way cycles is optimal. The finite number of types together with deterministic compatibility essentially creates a thick marketplace which does not explain why chains increase efficiency in practice. Our work deviates by studying a sparse marketplace with many agents that have a small probability of begin matched, where chains become an important factor.

Closest to our paper are Ashlagi et al. (2013), Anderson et al. (2013) and Akbarpour et al. (2014) who study mechanisms in dynamic matching markets with a stochastic underlying network where preferences are based on compatibilities. Ashlagi et al. (2013) study a class...
of batching policies in a finite-horizon model with hard and easy-to-match agents. They analyze the number of matches throughout the horizon. They find that when matching through 2-way and 3-way cycles, there is essentially no benefit in small batches over greedy policies, but chains lead to many more matches compared to 2-way or 3-way cycles. This paper focuses on agents’ average waiting time in an infinite horizon and we derive scaling laws for waiting time under various matching policies and thickness levels.

Akbarpour et al. (2014) study an infinite-horizon model with only hard-to-match agents and focus on bilateral matching policies. Agents in their model can depart and they focus on minimizing the loss rate. When agents departure times are known to the clearinghouse, a greedy policy will result in a significant loss. However, when agents’ departure times are unknown to the clearinghouse they find the greedy policy to be almost optimal, consistent with our results for the case of bilateral matchings and a market with only hard-to-match agents. This is due to the close relationship between their model and ours and the fact that in both formulations the objective translates to minimizing the expected number of agents waiting (see Section 4.2 for further discussion).

Anderson et al. (2013) study a special case of our model with only hard-to-match agents. They consider three settings of exchanges, 2-ways, 2 and 3-ways, and chains. In each of these settings, they find a greedy policy to be asymptotically optimal, and that moving from 2-ways or 3-ways to chains reduces significantly the average waiting time. There are two main differences between their work and ours. First, in their greedy policy that matches through a chain, they always search for the longest possible chain and find the average waiting time to be $\Theta\left(\frac{1}{p_H}\right)$. We find a myopic policy that advances the chain using only local information on the compatibility graph, which still leads to the same average waiting time (except for the case in which all agents are hard-to-match, in which case the waiting time increases by a small logarithmic factor). Second, and more importantly, our model allows to study what impact the presence of easy-to-match agents has on efficiency and on the market design. Recent studies have incorporated preferences that are not only based on compatibility. Baccara et al. (2015) study a dynamic two-sided matching market with two types of agents on each side. They find that in an optimal mechanism, agents wait for some period of time to be matched by their preferred type. They further find that a decentralized market is kidney exchange pools.

11 Their scaling is different than ours but closely related: every period a large number of agents arrive to the market and the probability to match is inversely proportional to the arrival rate. We model small arrival rate and hard-to-match agents match with a small probability.

12 This relationship is further formalized by Anderson et al. (2013) in a model with only hard-to-match agents.

13 Finding the longest chain is computationally hard, and requires global information on the whole compatibility graph.
inefficient. Fershtman and Pavan (2015) characterize the optimal mechanism in a many-to-many two-sided matching market in which agents’ preferences change endogenously over time. Doval (2014) and Kadam and Kotowski (2014) study stability in dynamic matching markets. Like Baccara et al. (2015), Doval (2014) finds that a clearinghouse should restrict agents’ waiting in order to achieve efficiency.\footnote{There is also a large literature concerned with search in labor markets, where firms and workers are randomly matched and decide whether to wait for better matches. See the survey by Rogerson et al. (2005).}

Another strand of research is about allocation in dynamic one-sided markets, and some papers in this thread study how to organize queues. Zenios (1999) analyzes a queueing model for the allocation of cadaver organs for patients with different types. Leshno (2014) designs queueing policies to minimize the mismatches of allocated objects, which translates into how to incentivize agents to wait for their preferred objects.\footnote{See also Bloch and Cantala (2014) for a study on a queueing system.} Note that a dynamic networked market can be viewed as a queue that serves itself.\footnote{Some papers that study optimal allocation mechanisms in static settings identify when organizing a queue is optimal (e.g., Hoppe et al. (2009) and Chakravarty and Kaplan (2013)).}

Finally, several papers study the timing of exchanges in markets with buyers and sellers. For example Mendelson (1982) studies the behavior of prices and quantities resulting from periodic trading. Budish et al. (2013) analyzes the tradeoff between the trading frequency and efficiency in continuous time financial markets.

\section{Setup}

We study an infinite-horizon dynamic matching market. For every two agents \(a\) and \(a'\), either \(a'\) is acceptable to \(a\) or not. We say that agent \(a\) can be matched by \(a'\) only if \(a'\) is acceptable to \(a\), in which case we also say that \(a'\) can match \(a\). There are two types of agents, hard-to-match and easy-to-match denoted by \(H\) and \(E\).\footnote{Note that in the Kidney exchange application, the patient and the donor together are considered to be one agent.} Any agent is acceptable to an \(H\) agent with probability \(p_H\) and to any \(E\) agent with probability \(p_E\), independently, where \(p_H < p_E\).

We will be interested in the case where \(p_H\) is significantly smaller than \(p_E\). In the context of kidney exchange, medical data determines whether the donor of a patient-donor pair \(a'\) can donate (or is acceptable) to the patient of the pair \(a\), and the two types may correspond to low and high sensitized patients, while abstracting away from blood types. In practice, the chance that a patient is incompatible with a random donor based on her antibodies follows approximately a bimodal distribution in kidney exchange programs (Ashlagi et al. (2012)).\footnote{This probability is called the Panel Reactive Antibody (PRA). A significant fraction of the population have either PRA above 97% or below 20%.}
Every period a new agent arrives to the market, whose type is $H$ or $E$ with probability $\theta$ and $1-\theta$, respectively. We assume that every agent is indifferent between acceptable matches and prefers to be matched as soon as possible and do not discount the future. Agents in our model leave the market only after they are matched. We therefore adopt the average waiting time of agents as a measure for efficiency.

We study matching policies in two different settings distinguished by how agents can get matched: bilaterally or indirectly through chains. In a bilateral match two agents match each other (which is the common form in the matching and search literatures). In a chain an agent $a$ who gets matched by some agent $a'$, matches a different agent $a''$. It is worth noting that in kidney exchange, chains initiated by an altruistic donor (who does not expect a kidney in return), account for the majority of transplants (Anderson et al. (2015)).

Assuming that the market reaches a steady-state distribution, we denote by $w_H$ and $w_E$ the average waiting times of $H$ and $E$ agents, respectively. In our analysis we use the observation that $w_H = n_H/\theta$, where $n_H$ is the average number of $H$ agents in steady-state (Little’s law).

Several assumptions require some discussion. Indifference between acceptable matches removes agents’ incentives to wait for preferable matches. This allows us, however, to focus on the tradeoffs between thickness, efficiency, and the matching technology. Assuming agents have a noisy set of potential matches is realistic. In kidney exchange, for example, there is large heterogeneity among sensitized patients’ antibodies, which determine their compatibility with donors. The stochastic structure can further capture agents’ tastes. Assuming that match probabilities are independent allows us to track the state of the market and dealing with correlated matches is very challenging. We believe most of our insights would hold in markets with more than two-types but a rigorous analysis is challenging.

Assuming that agents do not leave without matching ignores some interesting tradeoffs but enables us to focus on average waiting times. However, as discussed in Section 4.2, improving the average waiting time results in increasing the chance of matching before departure. Roughly speaking, this suggests that in a market with departures, the average waiting time and the match rate are aligned. Finally, fixing the arrival rate to be deterministic simplifies the analysis, and we believe our results still hold for Poisson arrival rates.

We use the following notations throughout the paper. Consider any two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}_{>0}$. We write $f = o(g)$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$; $f = O(g)$ if there exist a constant $a$ such that $f(n) \leq ag(n)$ for every sufficiently large $n$; $f = \Omega(g)$ if there exist a constant $b$ such that $bg(n) \leq f(n)$ for every sufficiently large $n$, and $f = \Theta(g)$ if $f = \Omega(g)$ and $f = o(g)$. 
3 Myopic matching and beyond

In this section, we first analyze myopic matching policies under bilateral matchings and matching through chains. Next we discuss the optimality of our findings in a broader context of non-myopic matching policies.

3.1 Bilateral matching

We study here myopic matching policies that allow only bilateral matches between pairs of agents. Our benchmark matching policy attempts to match each agent upon arrival bilaterally with another agent in the market while breaking ties in favor of hard-to-match agents. Formally, call BilateralMatch the policy that attempts to match an arriving agent $a$ upon arrival with an $H$ agent if possible (breaking ties randomly). If no $H$ agent can form a bilateral match with $a$, then $a$ is matched to an $E$ agent if possible, and otherwise $a$ remains in the market and waits for an eventual match.

The dynamic system that results from both the stochastic arrivals and the BilateralMatch policy can be represented as a Markov chain. Intuitively, agents that are waiting in the market never match each other. Therefore the number of $H$ and $E$ agents uniquely defines the state of the market at any time.

Next, we quantify the average waiting times in steady-state of $H$ and $E$ agents under BilateralMatch and find that $\theta = 0.5$ is a sharp threshold for the behavior of hard-to-match agents’ average waiting time.

**Theorem 1.** Under BilateralMatch, the market reaches steady-state. Furthermore, there exist positive constants $A_\theta$, $B_\theta$, and $C_\theta$ such that:

1. If $\theta < 1/2$, then
   \[ w_H \leq \frac{A_\theta}{p_E p_H}. \]  
   (1)

2. If $\theta \geq 1/2$, then
   \[ w_H \leq \frac{B_\theta}{p_E^2}. \]  
   (2)

3. For $0 \leq \theta < 1$
   \[ w_E \leq \frac{C_\theta}{p_E^2}. \]  
   (3)

Propositions 3 and 4 will show that (1) and (2) are in fact asymptotically tight.\(^{19}\) This shows that $H$ agents wait on average significantly less when they are on the short side

\(^{19}\)See also Appendix A.1 where we further discuss tightness, and provide simulations results which show that we can find exact values for $A_\theta$ and $B_\theta$. 

8
(θ < 0.5) than when they are on the long side (θ > 0.5) of the market. Hence, despite the fact that \( p_H \) is significantly smaller than \( p_E \), the balance between \( H \) and \( E \) agents in the market plays a key role in the average waiting times of \( H \) agents. The proof of Theorem 1 provides numerical values for the constants \( A_\theta, B_\theta, \) and \( C_\theta \).

Intuitively, agents’ average waiting time is inversely proportional to the probability for a match to occur. Under a myopic bilateral policy, no existing pair of agents in the market can match each other (otherwise they would match and leave the market). An arriving \( E \) agent forms a bilateral match with an existing \( E \) agent with probability \( p_E^2 \), implying that \( E \) agents experience relatively small waiting times, compared to what \( H \) agents experience. For an \( H \) agent, however, the probability of matching to an existing \( E \) agent is \( p_E p_H \) and to an existing \( H \) agent is \( p_H^2 \). When \( \theta < 0.5 \), we show that almost all \( H \) agents are matched with \( E \) agents, and thus their average waiting time is \( \Theta(1/p_E p_H) \). When \( H \) agents arrive more frequently than \( E \) agents, there are simply not enough \( E \) agents to match with \( H \) ones. This means that a non-negligible fraction of \( H \) agents must match with each other, thus increasing their average waiting time to \( \Theta(1/p_H^2) \).

Somewhat surprisingly, we find that when \( p_E = 1 \), Theorem 1 remains true even if the BilateralMatch policy is modified to break ties in favor of easy-to-match agents. We call this policy BilateralMatch(E).

**Proposition 1.** Let \( p_E = 1 \). Under the BilateralMatch(E), the market reaches steady-state, and there exist positive constants \( \tilde{B}_\theta, \tilde{C}_\theta \) such that:

1. if \( \theta < 1/2 \), then
   \[
   w_H \leq \frac{\tilde{A}_\theta}{p_H}.
   \]

2. if \( \theta \geq 1/2 \), then
   \[
   w_H \leq \frac{\tilde{B}_\theta}{p_H^2}.
   \]

The proofs of Theorem 1 and Proposition 1 rely on analyzing a Markov chain whose state space can almost be reduced to a single dimension that accounts for the number of \( H \) agents in the market. In fact, when \( p_E = 1 \), the state space is two dimensional, where one dimension has two possible states (the number of \( L \) agents in the market - zero or one). For the general case where \( p_E < 1 \) within the BilateralMatch(E) setting, or alternatively when ties are broken randomly among all agents, analyzing the state space is technically challenging, yet we conjecture that the same result holds with perhaps different constants.
3.2 Matching through chains

When only bilateral matchings can be formed we found that when $H$ agents are on the long side their average waiting time is large due to the rare coincidence of wants. In some marketplaces, however, agents match indirectly through chains. In kidney exchange a patient may receive a transplant from a donor of an incompatible pair $a$, while her intended donor will give a kidney to a patient of a different incompatible pair $a'$. In housing markets, agents typically purchase from one agent and sell to another or vice versa.\footnote{The timing of transactions may depend on the supply and demand in these markets.}

For our chain model, we introduce the notion of a \textit{bridge} agent, who has already been matched by some other agent, but is still waiting to match a different agent. We restrict attention to the case in which a single bridge agent arrives to the market at time $t = 0$ and therefore only a single chain exists at any given time.

Denote by ChainMatch the myopic policy under which the bridge agent $b$ attempts to match the incoming agent, $a_1$. If such a match is possible a chain segment is initiated in the current period as follows. We pick an agent $a_2$ uniformly at random among all other agents that $a_1$ can match (while prioritizing $H$ agents), and if such an agent exists, we pick uniformly at random an agent $a_3$ among the remaining agents that $a_2$ can match, and so forth. This process is repeated until the first time an agent, say $a_k$, cannot match any other agent in the market except $\{b, a_1, a_2, \ldots a_{k-1}\}$. We implement these matches (i.e., agents $b, a_1, \ldots, a_{k-1}$ leave the market), and $a_k$ becomes the new bridge agent. Note that under this matching policy there is always a single bridge agent in the market. Further notice that since matches are formed myopically, the policy does not necessarily implement the longest chain segment in each period.

For our next results, to simplify notations and technical challenges, we restrict our attention to the case in which $p_E = 1$, implying that $E$ agents never remain in the market.\footnote{This assumption allows us to analyze a single dimension Markov Chain.}

We believe that dropping this assumption is not crucial for our next result.\footnote{This belief is backed by the simulations reported in Appendix A.2.}

Theorem 2. Suppose that $p_E = 1$ and $0 < \theta \leq 1$. Under the ChainMatch policy, the market reaches steady-state and:

1. If $\theta < 1$, there exists a constant $K_\theta$ such that:
   \[ w_H \leq \frac{K_\theta}{p_H}. \]  
   (4)

2. If $\theta = 1$:
   \[ w_H \leq \frac{2\ln(1/p_H)}{p_H}. \]  
   (5)
The formal proof of Theorem 2 is given in appendix E. The main idea is to analyze the length of a new chain segment that is formed by ChainMatch. In Proposition 2 below we use the steady-state property of the system to compute the average length of chain segments. Further, we show that if the number of agents in the system becomes large (of the order of $\Theta\left(\frac{\ln(1/p_H)}{p_H}\right)$), the new chain segment will match a large fraction of agents. Together with Proposition 2 this allows us to upper-bound the number of agents in steady-state.

Figure 1 plots simulation results that compare the average waiting time of hard-to-match agents under the BilateralMatch and ChainMatch policies for a variety of thickness levels when $p_H = 0.02$ and $p_E = 1$.23 Observe that when $\theta$ takes values below 0.5 the difference between BilateralMatch and ChainMatch is small. Furthermore, note the threshold effect around $\theta = 0.5$, above which the average waiting time increases rapidly when bilateral matches are conducted.

Figure 1: The average waiting time for $H$ agents under the BilateralMatch and ChainMatch policies for $p_H = 0.02$ and $p_E = 1$.

Anderson et al. (2013) study a special case of our model with only hard-to-match agents (i.e., $\theta = 1$). They show that with a greedy chain policy that always selects the longest chain segment in the market, the average waiting time is $\Theta\left(\frac{1}{p_H}\right)$. The disadvantages of such a policy is that finding the longest chain segment is computationally hard and is only meaningful when it is implemented by a centralized planner. Our myopic approach is not only computationally straightforward, it can be implemented in a decentralized market as

23 For all the empirical results, we run 5 independent simulations, each for 100,000 time periods and to avoid transient states we only compute the average waiting times for agents that arrive in the last 20,000 periods.
well. Theorem 2 together with Proposition 3 show that using the myopic ChainMatch policy (as opposed to finding the longest chain segment) has little impact on $H$ agents’ average waiting time. When $\theta < 1$, $H$ agents will also wait on average $\Theta(\frac{1}{p_H})$ periods, and when all agents are hard-to-match, their average waiting time increases at most by a logarithmic factor.

The next result gives comparative statics for the expected length of a chain segment.

**Proposition 2.** Let $L_\infty$ be the length of a new chain segment formed by ChainMatch policy in steady-state, and suppose $p_E = 1$. Then we have:

$$E[L_\infty] = \frac{\theta(1 - p_H)}{\theta p_H + (1 - \theta)} + 1.$$ 

Proposition 2 implies that increasing arrival rates of easy-to-match agents decreases the expected length of a chain segment, which may reduce some logistical difficulties that arise with long chain segments. For example, in kidney exchange crossmatch (tissue-type) tests should pass prior to actual transplants, and in housing markets, buyers or sellers may wait for earlier transactions to take place.

The proof is given in appendix E.2. Intuitively, if a policy reaches steady-state, the number of agents matched in a chain segment corresponds to the number of agents that have arrived since the previous chain segment. Therefore, the expected length of a chain segment is the inverse of the frequency at which it occurs. Under the ChainMatch policy a chain segment begins when an arriving agent can be matched by the bridge agent, which happens with probability $\theta p_H + (1 - \theta)$, and in the meantime $\theta(1 - p_H)$ agents accumulate in the market.

### 3.3 Beyond myopic matching

So far we analyzed myopic matching policies. It is natural to ask whether one can decrease agents’ average waiting times by using more sophisticated matching policies.

The next result gives a lower bound on the average waiting time for any matching policy regardless of the matching technology.

**Proposition 3.** For any $p_E \leq 1$ and under any matching policy that reaches steady-state, for any $\theta \in (0,1)$ and any $p_H > 0$, there exists a constant $c_\theta$ such that $w_H + w_E \geq \frac{c_\theta}{p_H}$.

The proof uses similar ideas as in Anderson et al. (2013), who study the same model with only hard-to-match agents. We assume that there is at most one bridge donor in the system. The main intuition behind this proof is the following: Suppose that the pool size is

\[\text{The proof reveals that Proposition 2 holds true for any greedy chain policy that reaches steady-state.}\]
too small, then an arriving agent has a small probability of being matched immediately, and therefore must wait a “long” time to obtain at least one incoming edge. This long waiting time contradicts the small pool size (with Little’s law).\footnote{Therefore, having many chains running at the same time is the only way to reduce waiting times beyond $1/\rho_H$. Suppose that we have $\Omega(1/\rho_H)$ chains in the system. Then the probability of being matched immediately is very large, and waiting times become very small.}

Proposition 3 provides a bound on the sum of average waiting times. Therefore one may possibly decrease the average waiting time of hard-to-match agents beyond $1/p_H$ (for example, by letting easy-to-match agents wait and artificially thicken the market, one could increase $H$ agents’ chances to match and thus reduce their waiting times). This, however, comes at a cost of imposing high waiting times for easy-to-match agents. In particular, Proposition 3 together with results from 3.2 and 3.1 imply that when $\theta < \frac{1}{2}$, a myopic policy, whether using chains or bilateral matches, is optimal up to constant factors. When $\frac{1}{2} < \theta < 1$, matching myopically using chains is optimal up to constant factors.

It is natural to ask whether there exist a bilateral matching policy that decreases average waiting of hard-to-match agents when $\theta > 1/2$? The next proposition shows that the answer to this question is essentially negative.

**Proposition 4.** Let $\theta > 1/2$. For any bilateral matching policy that reaches a steady-state, under a mild regular condition, there exists a positive constant $c_\theta$ such that $w_H \geq c_\theta \frac{\rho}{\rho_H}$\footnote{The regular condition states that for every $H$ agent that arrives at steady-state the probability to eventually being matching to an $E$ agent is the same.}

## 4 Extensions

In this section, we consider four independent extensions to our model. In 4.1 the assumption that one agent, $H$ or $E$, arrives every period is relaxed. In 4.2 we discuss the implications from adding departures to our model. In 4.3, we discuss two-sided matching markets. Finally, in 4.4 we discuss our model in a decentralized environment.

### 4.1 The impact of thickening the pool

In our model every period a single agent arrives to the market, implying that the arrival rates of hard and easy-to-match agents are perfectly correlated. We relax this assumption in order to explore the effect thickening the market has on efficiency.

We conduct simulations assuming independent arrival rates $\theta_H$ and $\theta_E$ for $H$ and $E$ agents, respectively. In particular arrivals of $H$ and $E$ agents are modeled by two independent Bernoulli Processes with rates $\theta_H$ and $\theta_E$, respectively. For all our simulations we run 5...
independent trials, each for 100k time periods and to avoid transient states we only compute
the average waiting times for agents that arrive in the last 20,000 periods.

Simulation results suggest that our theoretical findings, which were derived under a
slightly different arrivals model, hold to a large extent. Figures 2 and 3 illustrate the behav-
or of the marketplace under the ChainMatch(H) and BilateralMatch(H) matching policies,
respectively, under a variety of arrival rates.

![Figure 2: Simulations of a pool with independent arrivals of H and E agents, using Chain-
Match and parameters $p_H = 0.02, p_E = 0.8$. Left panel shows the waiting times of H agents
as a function of $\theta_H$, for various values of $\theta_E$. Right panel shows waiting times for H agents
as a function of $\theta_E$, for various values of $\theta_H$.]

Observe that under the ChainMatch policy (Figure 2), the average waiting time for H agents always decreases with both $\theta_H$ and $\theta_E$. This means that attracting agents to join the market is always beneficial.

Interestingly, the average waiting time for H agents is not monotone under the Bilateral
Match policy (left panel in Figure 3). Observe that for large values of $\theta_E$, the average waiting
time for H increases with $\theta_H$. In line with our theoretical findings, this increase is steeper
when $\theta_H \geq \theta_E$. However, for small values of $\theta_E$, after $\theta_H$ becomes large enough, the average
waiting time decreases again (each of these plots will decrease after $\theta_H$ becomes sufficiently
large). Intuitively due to a shortage of E agents in this regime, many H agents match with
each other. The right panel shows that for any value of $\theta_H$, the average waiting time of H agents decreases with $\theta_E$. This decrease is sharp as long as $\theta_E \leq \theta_H$.

So when matches are formed through chains, thickening the market by increasing either
$\theta_E$ and $\theta_H$ always reduces waiting times. Under bilateral matching, however, two phenomena
arise. If possible, increasing the arrival of easy-to-match agents to the level where $\theta_E \geq \theta_H$

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27We find similar results when ChainMatch(E) and BilateralMatch(E) are tested.
Figure 3: Simulations of a pool with independent arrivals of $H$ and $E$ agents, using *BilateralMatch* and parameters $p_H = 0.02$, $p_E = 0.8$. Left panel shows the waiting times of $H$ agents as a function of $\theta_H$, for varying $\theta_E$. Right panel shows waiting times for $H$ agents as a function of $\theta_E$, for varying $\theta_H$.

will significantly reduce waiting times of $H$ agents. Otherwise, when $\theta_H \gg \theta_E$, one can reduce average waiting times by increasing the arrivals of hard-to-match agents to the market.

We remark here that the average waiting time of $E$ agents is orders of magnitude lower than that of $H$ agents (we report these average waiting times in Appendix [C]).

### 4.2 Departures

In our model everyone agents leaves the market matched. We briefly discuss how the results change when departure rates are incorporated into the model, allowing agents to leave the market unmatched. We argue heuristically that minimizing the average waiting time in our model with no departures is closely related to minimizing the likelihood that an agent will depart before being matched in a model with departures. Roughly speaking the idea is that minimizing the expected waiting time in our model is the same as minimizing the expected number of agents in the market (by Little’s law), which in turn reduces the rate of agents that depart the market without being matched.\footnote{This relation is formalized by Anderson et al. (2013) in a model with only hard-to-match agents.}

Consider a dynamic matching market with $\theta < 1$ and suppose that at each period every unmatched $H$ agent departs the market independently with probability $q_H$.\footnote{Since easy-to-match agents rarely wait to be matched, we simplify the discussion by ignoring their departure rate.} In steady-state the equilibrium between arrivals and departure rates yields

$$\theta = \mu + q_H n_H,$$ 

\footnote{This relation is formalized by Anderson et al. (2013) in a model with only hard-to-match agents.}
where $n_H$ is the expected number of $H$ agents in the market and $\mu$ is the expected match rate. By Little’s law, the average total waiting time for $H$ agents (over both matched and unmatched agents) is $w_H = \frac{n_H \mu}{\theta}$, implying that in steady-state

$$w_H = \frac{1 - \mu/\theta}{q_H} = O(1/q_H).$$ (7)

Therefore reducing agents’ average total waiting times is aligned with reducing departures

We can now discuss the effects different departures rates have on our results. Hard-to-match agents’ overall average waiting time (both matched and unmatched) will remain the same as in the original model under myopic policies if their departure rate scales similar to their average waiting time in the model without departures. Under this regime the only difference is that a constant fraction of hard-to-match agents will leave the market unmatched.

Recall that in our original model agents’ average waiting time is $\Theta(1/p_H)$ both under the $\text{ChainMatch}$ policy (for any $\theta < 1$) and under the $\text{BilateralMatch}$ policy for $\theta < 0.5$ (note that in these settings only when $q_H = \Theta(p_H)$ the likelihood that an agent will depart before being matched is bounded away from 0 and 1).

An interesting effect happens when $\theta > 0.5$ and agents match bilaterally. Note that under the $\text{BilateralMatch}$ policy, the average number of $H$ agents in the market $\min\{\Theta(1/p_H^2), O(1/q_H)\}$. When the departure rate is $q_H = o(p_H^2)$ (but at least the order of $p_H$) the market behaves essentially as a bipartite market: since the probability that two $H$ agents match each other is very small, almost all $H$ agents are either matched with $E$ agents, or depart before they have a chance to match with other $H$ agents.

### 4.3 Two-sided matching markets with departures

Many matching markets are naturally two-sided and we discuss how to extend our analysis to such markets with departure rates. Consider a two sided market in which every period one agent arrives to the market. Denote the two sides by $S$ and $L$ and let $\theta > 0.5$ be the probability that an arriving agent belongs to $L$. Since agents on the long side will accumulate (they cannot match with each other), it is natural to incorporate a small departure rate $q_L$ for $L$ agents which will allow the system to reach a steady-state.

Denote by $p_{\text{match}}$ the probability that a given a pair of agents, one from each side, can form a bilateral match. One can show that under the $\text{BilateralMatch}$ policy, the overall

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30 Analyzing the tradeoff between match rates and the waiting times for matched agents remains an interesting open question.

31 Note that $q_H = \theta(p_H^2)$ scales similar to the waiting time in the model without departures and any other scaling leads to trivial probabilities for departing prior to being matched.
average waiting time for $L$ agents (both matched and not) is $\Theta(1/q_L)$, independently of $p_{\text{match}}$. The idea is that for all $p_{\text{match}}$ and $q_L$, a significant fraction of $L$ agents will depart the market without being matched and these agents’ waiting time is of order $1/q_L$.

Let $N_L$ be the number of agents in the market at a given time and suppose $N_L = \Theta(1/q_L)$. Then an arriving $S$ agent will match immediately with very high prob Then an incoming $S$ agent matches immediately with probability $1 - (1 - p_{\text{match}})^N_L \approx 1 - e^{-p_{\text{match}}/q_L} + \sigma(p_{\text{match}}/q_L)^3$. So for $q_L$ that is significantly smaller than $p_{\text{match}}$, $S$ agents match upon arrival with high probability and a fraction $1 - \theta$ of $L$ agents leave the market matched.\(^{33}\)

One interesting implication is that a chain in a two-sided market would not lead to better waiting times than bilateral matchings since agents on the short side match immediately upon arrival and agents on the long side cannot match each other.

Observe that the two-sided model has striking similarities to the one-sided model we studied earlier. In both models, there is a parameter which allows the long side to self-regulate: $q_L$ through departures in two-sided markets, or $p_H^2$ for $H$ to $H$ matches in one-sided markets. When $\theta > 0.5$, this parameter determines the $H$ agents’ waiting times. We therefore believe that similar results to the one sided model will hold in two-sided markets.\(^{34}\)

### 4.4 Decentralized matching

Consider a decentralized version of our model, in which agents that join the market form matches selfishly. Note that since preferences are based on compatibility, agents will behave myopically in equilibrium. Therefore, with some caution one may apply our findings regarding the average waiting times under myopic matching policies to decentralized markets.

Suppose first that only bilateral matches can be formed and $p_E = 1$. Since the average waiting time of $H$ agents is approximately the same under $\text{BilateralMatch}(H)$ (Theorem 1) and $\text{BilateralMatch}(E)$ (Proposition 1), their average waiting time in a decentralized market is also the same. One shortcoming of this conclusion is that in a decentralized market agents are likely to break ties arbitrarily among compatible matches without assigning priority to

\(^{32}\)For example when $q_L < p_{\text{match}}/5$, the chance that an $S$ agent matches immediately is approximately $1 - e^{-5} \approx 99.4\%$.

\(^{33}\)Thus when $q_L$ is small, departure rates for $S$ agents may be ignored as they do not accumulate in the market. Note, however, that whether agents are on the long or side depends on departure and arrival rates of both sides.

\(^{34}\)One interesting application is a sub-market in kidney exchange with O-A (O patients and A donors) and A-O incompatible pairs. A-O pairs are on the short side in this market since many A patients are compatible to their live O indented donor and thus do not enter this market. A-O pairs in kidney exchange are likely to be much more sensitized than patients in O-A pairs since these A-O pairs must be tissue type incompatible. Our findings show that pairs on the short side are easy-to-match in practice despite their patients’ high sensitivity while O-A pairs are under-demanded and will find it difficult to match.
either type. We believe that breaking ties randomly will not change this conclusion.

Figure 4 reports simulation results for the *BilateralMatch* policy when ties are broken either in favor of *E* agents or in favor of *H* agents. While the results are given only for $\theta = 0.4$ and $\theta = 0.6$, we observed a small gap in the average waiting times for $\theta < 0.5$, and otherwise there is essentially no gap. This is intuitive since when $\theta > 0.5$ almost all *E* agents match to *H* agents upon arrival.

![Figure 4: Average waiting times for *H* agents under *BilateralMatch* policies with parameters $p_H = 0.02$ and $p_E = 0.8$. We compare policies that either break ties in favor of *E* agents or in favor of *H* agents.](image)

Figure 5: Average waiting times for *H* agents under *ChainMatch* policies with parameters $p_H = 0.02$ and $p_E = 0.8$. We compare policies that either break ties in favor of *E* agents or in favor of *H* agents.

When matches are formed indirectly through a chain, Theorem 2 provides the equilibrium average waiting time of hard-to-match agents in a decentralized market when $p_E = 1$. Figure
plots simulation results that compare the ChainMatch policy which break ties in favor of $H$ agents and a similar policy that break ties in favor of $E$ agents. Note that there is essentially no difference, since $E$ agents never remain in the market.

5 Conclusion

We studied a dynamic matching market and analyzed how the thickness and the forms of matchings (bilateral or through a chain) affect efficiency. We identified that the balance between hard and easy-to-match agents in the market plays a crucial role on the desired matching technology. In particular, matching through a chains (even a single one) leads to significantly better waiting times than bilateral matching only when hard-to-match agents arrive more frequently to the market. Moreover there is almost no harm from adopting myopic matching policies.

Our work has several implications. First, in a decentralized market, the equilibrium in which agents attempt to match as soon as possible, is asymptotically efficient. Second, the lack of thickness cannot be compensated by better matching mechanisms and therefore the only way to thicken the market fruitfully is by attracting more agents. Our findings depend strongly on the assumption that agents preferences are based on compatibility. Indeed Baccara et al. (2015) and Doval (2014) find that with more preference structure optimal matching policies are not myopic.

Our results imply that agents’ average waiting time depends crucially on whether they are on the long or short side of the market. In two-sided markets, agents on the short side of the market will match quickly even if their a priori chance to be matched by an arbitrary agent is very small. Therefore, while chains play a crucial role in one-sided markets, they have almost no benefit over bilateral matchings in two-sided markets.

This work raises several questions. Thickening the marketplace, by having more easy-to-match agents naturally increases the match rate. However, this rate increase depends on the composition of the market and is highly non-linear, raising the issue of how to attract different types of agents to the marketplace. Finally, what forms of exchanges will arise in a decentralized market is a question that relates to research on the origin of money (Jones 1976 and Kiyotaki and Wright 1989).

Our findings hold for a single chain and one can further reduce waiting times by having multiple chains. We believe, however, that any constant number of chains will have no asymptotic advantage over a single chain.

For instance, hospitals in the U.S. often conduct internal exchanges with easy-to-match pairs rather than enrolling them in to kidney exchange clearinghouses (Roth et al. 2007; Ashlagi and Roth 2013).

These papers study how patterns of trade, such as direct barter and using currency, arise in equilibrium.
References


### A Tightness of the theoretical bounds

In the next following subsections, we provide simulation results under the myopic *Bilateral-Match* and the *ChainMatch* policies and examine how tight are the theoretical upper bounds on the waiting times we derived.
A.1 Bilateral matching

Theorem 1 states that under the BilateralMatch policy, there exist constants $B_\theta$ and $C_\theta$ such that (i) $w_H \leq \frac{B_\theta}{p_{E|PH}}$ if $\theta < 1/2$, and (ii) $w_H \leq \frac{C_\theta}{p_H}$ if $\theta \geq 1/2$. The proof (Appendix D) yields the following numerical values for these constants:

$$A_\theta = \frac{\ln \left( \frac{1-\theta}{1-2\theta} \right)}{\theta} + \frac{2\theta}{\theta(1-2\theta)\ln \left( \frac{1-\theta}{1-2\theta} \right)} + 1,$$

$$B_\theta = \ln(2\theta) + 2/\ln(2\theta).$$

Simulations show that the upper bounds with these constants are not very tight and we believe that this is mainly due to an additional term needed for technical reasons. We conjecture that the average waiting time of hard-to-match agents in the market at steady-state are bounded according to (8) and (9).

- If $\theta < 1/2$, then

$$w_H \leq \frac{\ln \left( \frac{1-\theta}{1-2\theta} \right)}{\theta p_{E|PH}}.$$  \hspace{1cm} (8)

- If $\theta \geq 1/2$, then

$$w_H \leq \frac{\ln(2\theta)}{\theta p_H^2}.$$  \hspace{1cm} (9)

Simulations show that these conjectured upper bounds are indeed tight for values of $\theta$ that are sufficiently far enough from 0.5. However, as $\theta$ approaches 0.5, these bounds are tight only for sufficiently small $p_H$. Figure 6 shows simulation results alongside the conjectured bounds for $\theta = 0.1$ and $\theta = 0.6$.

A.2 Matching through chains

Theorem 2 states that under the ChainMatch policy there exists a constant $K_\theta$ such that (i) $w_H \leq \frac{K_\theta}{p_{PH}}$ when $\theta < 1$, and (ii) $w_H \leq \frac{2\ln(1/p_H)}{p_{PH}}$ when $\theta = 1$.

The proof (Appendix E) yields the following numerical value for $K_\theta$:

$$K_\theta = \frac{2}{(1-p_H)} + \frac{\theta(1-p_H)}{\theta p_H + (1-\theta)} + o(1).$$

Simulations show that the upper bounds with this constant are not tight and we conjecture that the tighter inequalities (10) and (11) hold true.

- If $\theta < 1$,

$$w_H \leq \frac{1}{2\theta p_H} \left( \frac{\theta}{1-p_H} + \frac{\theta(1-p_H)}{\theta p_H + (1-\theta)} \right).$$  \hspace{1cm} (10)
Figure 6: Numerical results (solid red) and the conjecture bound (dashed blue) for $\theta = 0.1$ and $\theta = 0.6$ under the BilateralMatch policy.

- If $\theta = 1$, 
  \[ w_H \leq \frac{\ln(1/p_H)}{\theta p_H}. \]  
  (11)

Figure 7 shows simulation results under the ChainMatch policy alongside the conjectured bounds. We note that the conjecture is tight when $\theta = 1$ or $\theta < 1/2$. However, for $1/2 < \theta < 1$, it seems that our conjectured bounds are not perfectly tight.

B Proof Notations

Throughout the appendices below we denote by $B(\lambda)$ a random variable drawn from a Bernoulli distribution with parameter $\lambda$. Similarly, $\text{Bin}(n, p)$ corresponds to a random variable drawn from a Binomial distribution, with parameters $n$ and $p$.

All the proofs for the existence of steady-state of the Markov chains that we consider rely on the following result:

**Proposition 5** (Foster (1953)). Suppose $\{X_k\}$ is a discrete time, irreducible Markov Chain on a countable state space $\mathcal{X}$. If there exists a function $V : \mathcal{X} \mapsto \mathbb{R}$, $> 0$, and a finite set $B \subset \mathcal{X}$ such that for all $x \in B$, 
\[ \mathbb{E}_x[V(X_1) - V(X_0)] < \infty. \]  
(12)

and for all $x \in \mathcal{X}\setminus B$, 
\[ \mathbb{E}_x[V(X_1) - V(X_0)] < -\gamma \]  
(13)

then $X_k$ is positive recurrent.
Figure 7: Numerical results (solid red) and the conjecture bound (dashed blue) under the ChainMatch policy.

C Lower bounds: proofs of Propositions 3 and 4

We prove the lower bound results stated in Section 3.3.

Proposition 3. For any $p_E \leq 1$ and under any matching policy that reaches steady-state, for any $\theta \in (0, 1)$ and any $p_H > 0$, there exists a constant $c_\theta$ such that $w_H + w_E \geq c_\theta \frac{p_H}{p_H}$.

Proof of Proposition 3. The main intuition behind this proof is the following: Suppose that the pool size is too small, then an arriving agent has to wait a “long” time to obtain at least one incoming edge. This long waiting time contradicts the small pool size (with Little’s law).

Let $n_H$ and $n_E$ be respectively the expected numbers of $H$ and $E$ agents in the market in steady-state, and let $n = n_H + n_E$. Little’s law implies that $w_H = n_H/\theta$ and $w_E = n_E/(1-\theta)$. Therefore, it is enough to prove that there exists a constant $k_\theta$ such that $n \geq k_\theta/p_H$ (we then choose $c_\theta = \frac{k_\theta}{\max(\theta, 1-\theta)}$).

Let $k_\theta$ be a constant to be defined later. Assume for contradiction that there exists $p_H$
such that $n < k_\theta/p_H$. Let $i$ be an $H$ agent entering the market at steady-state, and let $W_i$ be her waiting time until she is matched. Let $\mathcal{V}$ be the set of agents in the market when agent $i$ arrives. Note that $\mathbb{E}[|\mathcal{V}|] = n \leq k_\theta/p_H$. Define the event $E_1 = \{|\mathcal{V}| \leq 3n/\theta\}$. By Markov’s inequality and Little’s law, $\mathbb{P}[E_1] \geq 1 - \frac{\mathbb{E}[|\mathcal{V}|\theta]}{3n} \geq 1 - \theta/3$.

Let $\mathcal{A}$ be the first $3n/\theta$ arrivals after $i$, and let $E_2$ be the event that at least one agent from $\mathcal{V} \cup \mathcal{A}$ has an outgoing edge towards $i$. We have

$$\mathbb{P}[E_2] = \mathbb{P}[\text{Bin}(|\mathcal{V}| + |\mathcal{A}|, p_H) \geq 1]$$

Therefore we get:

$$\mathbb{P}[E_2 \mid E_1] \leq \mathbb{P}[\text{Bin}(6n/\theta, p_H) \geq 1] \leq \mathbb{P}[\text{Bin}(6k_\theta/\theta p_H, p_H) \geq 1] \leq 6k_\theta/\theta.$$  

Where the first inequality derives from the definition of $E_1$, the second uses the fact that $n \leq k_\theta/p_H$ and the third is Markov’s inequality.

We now use the fact that if $i$ doesn’t have any edge from either $\mathcal{V}$ or $\mathcal{A}$, then she must wait longer than $3n/\theta$ time steps. Together with the bounds derived above, we get

$$w_H = \mathbb{E}[W_i] \geq \frac{3n}{\theta} \mathbb{P}[E_2] \geq \frac{3n}{\theta} \mathbb{P}[E_2 \mid E_1] \mathbb{P}[E_1] \geq \frac{3n}{\theta} (1 - 6k_\theta/\theta)(1 - \theta/3) \geq \frac{3n}{\theta} (1 - 6k_\theta/\theta)(2/3).$$

Thus we get:

$$n \geq n_H = w_H \theta \geq 2n(1 - \frac{6k_\theta}{\theta}).$$

Therefore for $k_\theta = \frac{\theta}{24}$, we obtain a contradiction.

Proposition 4. Let $\theta > 1/2$. For any bilateral matching policy that reaches a steady-state, under a mild regular condition, there exists a positive constant $c'_\theta$ such that $w_H \geq c'_\theta \frac{p_H}{p_H}$.  

**Proof of Proposition 4** Consider a bilateral matching policy $\mathcal{P}$, i.e. a policy that can only match pairs of agents. We allow any possibility for $\mathcal{P}$ regarding the choice of whether to match agents and which agents to match. The main idea is to show that a significant fraction of $H$ agents have to match to each other as a necessary condition for steady-state. Similarly to the proof of Proposition 3, we focus on finding a lower bound on the number of $H$ agents in steady-state.

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38Note that we abuse the notation of $\text{Bin}(n,p)$ by allowing its parameters to be random variables. In this case, conditional on the event $|\mathcal{V}| + |\mathcal{A}| = k$, the random variable $\text{Bin}(|\mathcal{V}| + |\mathcal{A}|, p)$ has a binomial distribution with parameters $k$ and $p$.  

39The regular condition states that for every $H$ agent that arrives at steady-state the probability to eventually being matching to an $E$ agent is the same.
Let $c_\theta$ be a constant to be defined later, and assume for contradiction that there exists $p_H$ such that $n_H \leq c_\theta/p^2_H$. Let us consider an $H$ agent entering the system at steady-state and let $W_i$ be its waiting time. Let $\mathcal{V}_H$ be the set of $H$ agents in the pool upon its arrival, which implies $\mathbb{E}[|\mathcal{V}_H|] = n_H \leq c_\theta/p^2_H$. We define the event $E_1 = \{|\mathcal{V}_H| \leq u_1^\theta n_H\}$ where $u_1^\theta$ is a function of $\theta$ to be defined later. By Markov’s inequality, $\mathbb{P}[E_1] \geq 1 - \frac{1}{u_1^\theta}$. Let $\mathcal{A}_H$ be the next $u_2^\theta n_H$ arrivals of $H$ agents, where $u_2^\theta$ is a function of $\theta$ to be defined later. We consider different matching scenarios.

1. The agent eventually gets matched to an $E$ agent. We call this event $E_2$.

2. The agent is matched to an agent from $\mathcal{V}_H \cup \mathcal{A}_H$. We call this event $E_3$.

3. The agent is matched to a later $H$ agent outside $\mathcal{V}_H \cup \mathcal{A}_H$. We call this event $E_4$.

We obtain:

$$\mathbb{P}[E_3] \leq \mathbb{P}[\text{Bin}(|\mathcal{V}_H| + |\mathcal{A}_H|, p^2_H) \geq 1], \quad (14)$$

We compute $\mathbb{P}[E_3]$ as follows:

$$\mathbb{P}[E_3|E_1] \leq \mathbb{P}[\text{Bin}((u_1^\theta + u_2^\theta)n_H, p^2_H) \geq 1] \leq \mathbb{P}[\text{Bin}((u_1^\theta + u_2^\theta)c_\theta/p^2_H, p^2_H) \geq 1] \leq (u_1^1 + u_2^\theta)c_\theta, \quad (15)$$

where we used the definition of $E_1$, the fact that $n_H \leq c/p^2_H$, and Markov’s inequality.

Note that because the system is in steady-state, any $H$ agent will be matched with probability 1. Furthermore, because the system has the fairness property, the probability of any $H$ agent to eventually match an $E$ is the same, which we denote $p_{H-E}$. Because the system is in steady-state, we have $\theta p_{H-E} = 1 - \theta$. Therefore:

$$\mathbb{P}[E_2] \leq \frac{1 - \theta}{\theta}. \quad (16)$$

Putting everything together, we get:

$$\frac{n_H}{\theta} = \mathbb{E}[W_i] \geq u_2^\theta n_H \mathbb{P}[E_4] \geq u_2^\theta n_H (\mathbb{P}[E_3^c] - \mathbb{P}[E_2])$$

$$\geq u_2^\theta n_H \left( \mathbb{P}[E_3^c|E_1] \mathbb{P}[E_1] - \frac{1 - \theta}{\theta} \right)$$

$$\geq u_2^\theta n_H \left( (1 - (u_1^\theta + u_2^\theta)c_\theta)(1 - \frac{1}{u_1^\theta}) - \frac{1 - \theta}{\theta} \right) \quad (17)$$

where we used Little’s law, then the fact that the expected waiting time conditional on $E_4$ is greater than $u_2^\theta n_H$, then the fact that $1 = \mathbb{P}[E_2] + \mathbb{P}[E_3] + \mathbb{P}[E_4]$, and (16).

Note that here it becomes clear that the assumption that $\theta > 1/2$ is necessary in order to obtain a contradiction (otherwise the right hand term is negative).
Here, because $\theta > 1/2$, we have: $\frac{1-\theta}{\theta} < 1$, and we can consider the mid-point $\alpha_\theta = \frac{1}{2} \left( \frac{1-\theta}{\theta} + 1 \right)$. Therefore taking for instance $u^1$ such that $1 - \frac{1}{u^1_\theta} = \sqrt{\alpha_\theta}$, $u^2_\theta = \frac{4}{2\theta-1}$, and $c_\theta$ such that $(1 - (u^1_\theta + u^2_\theta)c_\theta = \sqrt{\alpha_\theta}$, we get:

\[
\frac{n_H}{\theta} \geq \frac{u^2_\theta}{n_H} \left( (1 - (u^1_\theta + u^2_\theta)c_\theta)(1 - \frac{1}{u^1_\theta}) - \frac{1-\theta}{\theta} \right) \\
= \frac{u^2_\theta}{n_H} \left( \sqrt{\alpha_\theta} \sqrt{\alpha_\theta} - \frac{1-\theta}{\theta} \right) \\
= \frac{u^2_\theta}{n_H} \frac{1}{2} \left( 1 - \frac{1-\theta}{\theta} \right) \\
= \frac{4}{2\theta-1} \frac{n_H}{2\theta} \\
= \frac{2n_H}{\theta}
\]

Thus we obtain a contradiction. $\square$

\section{Bilateral matching: proof of Theorem 1}

**Theorem 1.** Under BilateralMatch, the market reaches steady-state. Furthermore, there exist positive constants $A_\theta$, $B_\theta$, and $C_\theta$ such that:

1. If $\theta < 1/2$, then
   \[
   w_H \leq \frac{A_\theta}{p_E P_H}.
   \]
2. If $\theta \geq 1/2$, then
   \[
   w_H \leq \frac{B_\theta}{p_H^2}.
   \]
3. For $0 \leq \theta < 1$
   \[
   w_E \leq \frac{C_\theta}{p_E^2}.
   \]

The proof involves several steps, which we will state using intermediate lemmas for the important results. The outline of the proof is as follows:

- We represent BilateralMatch as a two dimensional stochastic process $(N_{H,t}, N_{E,t})$ where $N_{H,t}$ $(N_{E,t})$ is the number of $H$ $(E)$ agents in the system at time $t$. We prove that this process is a Markov chain, which admits a steady-state distribution$(N_{H,\infty}, N_{E,\infty})$. Cf. Lemma 1 in Appendix D.1
- We define formally the waiting times $w_H$ and $w_E$, and use Little’s law to prove the results by focusing on the average steady-state number of agents $n_H$ and $n_E$.

- We couple our Markov chain with an auxiliary 1-dimensional Markov chain $M_t$ and show that $\mathbb{E}[N_{H,\infty}] \leq \mathbb{E}[M_\infty] + 1$. This allows us to focus on proving bounds on $M_t$.

- We provide three proofs using coupling techniques to obtain the bounds announced in equations (1), (2) and (3). Proofs can be found in appendix D.4 D.3 D.5

D.1 Markov chain representation

Observe that because the algorithm matches agents in a myopic way (i.e. matches are conducted whenever they are possible), there can never be two agents in the system that have the possibility to match to each other. Therefore, it is enough to only keep track of the number of $H$ and $E$ agents to completely characterize the system. At any given time $t$, $N_{H,t}$ ($N_{E,t}$) represents the number of $H$ ($E$) agents in the pool.

**Lemma 1.** For any $0 < p_H < 1$, $0 < p_E \leq 1$ and $0 < \theta \leq 1$, the number of agents in the pool under BilateralMatch $(N_{H,t}, N_{E,t})_{t \in \mathbb{N}}$ is a positive recurrent Markov chain and it converges to a steady-state distribution $(N_{H,\infty}, N_{E,\infty})$.

The proof is deferred to appendix D.6. This results allows us to apply Little’s law to our system, which yields $w_H = \mathbb{E}[N_{H,\infty}]/\theta$ and $w_E = \mathbb{E}[N_{E,\infty}]/(1 - \theta)$. In all the proof that follows, we will therefore focus on providing upper bounds for $\mathbb{E}[N_{H,\infty}]$ and $\mathbb{E}[N_{E,\infty}]$.

The structure of this Markov chain is shown in figure 8, and the transitions probabilities are in equation (28). Analyzing 2-dimension Markov chains can be very difficult, therefore we use a coupling technique to gain analytic tractability.

D.2 Coupling to a 1-dimensional Birth-Death process.

We now introduce an auxiliary Markov chain $M_t$ defined by:

$$M_{t+1} = \begin{cases} (M_t + 1) \text{ with probability } \theta(1 - p_H^2)^{M_t}, \\ (M_t - 1) \text{ with probability } \theta(1 - (1 - p_H^2)^{M_t}) + (1 - \theta)(1 - (1 - p_E p_H)^{M_t}), \\ (M_t) \text{ with probability } (1 - \theta)(1 - p_E p_H)^{M_t}. \end{cases}$$

(19)

**Remark.** This corresponds to the number of $H$ agents in the pool under the following “simplified” algorithm: For each incoming agent $v$ to the pool, match $v$ to one of the $H$ agents in the pool if such a match exists. If no match is found, add $v$ to the pool if it is of type $H$, otherwise discard $v$. Note that we do not keep unmatched $E$ agents, which is why we get a
1-dimensional Markov Chain, and why we expect the number of unmatched $H$ agents to be larger in this auxiliary chain.

The next lemma states that with an appropriate coupling, $N_{H,t}$ can be upper bounded by $M_t$. This allows us to focus on analyzing for $M_t$.

**Lemma 2.** The Markov chain $M_t$ defined in (19) reaches a steady-state distribution $M_\infty$, and there exists a way to couple the Markov chains $(N_{H,t}, N_{H,t})$ and $M_t$ such that for every $t > 0, N_{H,t} \leq M_t + 1$. Therefore $E[N_{H,\infty}] \leq E[M_\infty] + 1$.

The proof of lemma 2 is deferred to Appendix D.6.

**D.3 Proof of Theorem 1 part 1**

**Proof.** In this regime there are more $E$ agents than $H$ agents, and therefore we expect most of the $H$ agents to eventually get matched to $E$ agents. Therefore, we couple $M_t$ to a simple Birth-Death process that corresponds to the following algorithm: We never try to match arriving $H$ agents (this is unlikely anyways because we never keep $E$ agents in the system). We only try to match incoming $E$ agents to a fixed number of $H$ agents, equal to $\eta p_{E|PH}$ agents for $\eta = \ln(\frac{1-\theta}{1-2\theta})$. This will lead to a new Markov chain $S_t$ for which we can effectively compute the expected number of agents.

We can compute the transition states:

\[
S_{t+1} = \begin{cases} 
(S_t + 1) \text{ with probability } \theta, \\
(S_t - 1) \text{ with probability } (1 - \theta)(1 - (1 - p_{EPH})^{\frac{\eta}{p_{EPH}}}), \\
(S_t) \text{ with probability } (1 - \theta)(1 - p_{EPH})^{\frac{\eta}{p_{EPH}}}. 
\end{cases}
\]  

(20)

In the following lemma, we show that the chain $S_t$ reaches steady-state and we provide a bound on the expected number of $H$ agents in steady-state.

**Lemma 3.** $S_t$ reaches steady-state $S_\infty$ and we have

\[
E[S_\infty] = \frac{2\theta}{(1 - 2\theta) \ln \frac{1-\theta}{1-2\theta} p_{EPH}} + o(1/p_H).
\]  

(21)

The proof can be found in Appendix D.6.3

We now wish to couple the random walk $S_t$ with the Markov chain $M_t$ defined in (19).

**Lemma 4.** There exists a coupling of $M_t$ and $S_t$ such that for all $M_t > \frac{\eta}{p_{EPH}}$ we have:

\[M_t - M_{t-1} \leq S_t - S_{t-1}\]
The proof of Lemma 4 is deferred to the Appendix D.6, but the intuition is that if there are more than $\frac{n}{p_{EPH}}$ agents in the system, then $S_t$ grows faster than $M_t$ because it only matches agents "up to" the first $\frac{n}{p_{EPH}}$. Using this lemma, for all $t$ let us denote $t^*(t) = \max\{u \in \mathbb{R} \text{ s.t. } M_{u-1} \leq \frac{n}{p_{EPH}}\}$ we get

$$M_t \leq M_{t^*-1} + (S_t - S_{t^*-1}) \leq \frac{n}{p_{EPH}} + S_t.$$ 

We get:

$$\mathbb{E}[N_{H,\infty}] \leq \mathbb{E}[M_\infty] + 1 \leq \frac{n}{p_{EPH}} + \mathbb{E}[S_\infty] + 1 \leq \frac{\ln \left(\frac{1-\theta}{1-2\theta}\right)}{p_{EPH}} + \frac{2\theta}{(1-2\theta)\ln \left(\frac{1-\theta}{1-2\theta}\right)} + 1. \quad (22)$$

Where

$$A_\theta = \ln \left(\frac{1-\theta}{1-2\theta}\right) + \frac{2\theta}{(1-2\theta)\ln \left(\frac{1-\theta}{1-2\theta}\right)} + 1. \quad (23)$$

Remark. We note that using the exact same proof it is possible to derive another bound, which in some cases is tighter. By letting $\eta = \ln(\frac{1-\theta}{1-2\theta})$ with $0 < \delta < \theta$ we get another constant:

$$A_\theta = \ln \left(\frac{1-\theta}{1-2\theta - \delta}\right) + \frac{2\theta}{\delta + o(1)} + 1 = \ln \left(\frac{1-\theta}{1-2\theta - \delta}\right) + o(1).$$

In Section A.1, simulations show that $\ln \left(\frac{1-\theta}{1-2\theta}\right) / p_{EPH}$ seems to be a tight constant. However, proving tightness remains an open question.

D.4 Proof of Theorem 1 part 2

Proof. The proof is similar to the previous case. Again, we couple the Markov chain $M_t$ with a simpler random walk that is the result of the following algorithm: we only try to match incoming agents ($E$ and $H$) to a "virtual" set of $\ln(2\theta)/p_{EH}$ $H$ agents. This leads to a random walk $S'_t$ that provides an upper bound on $M_t$ when the number of agents becomes large:

$$S'_{t+1} = \begin{cases} 
(S'_t + 1) \text{ w.p. } \theta g(p_H), \\
(S'_t - 1) \text{ w.p. } \theta(1 - g(p_H)) + (1 - \theta)(1 - f(p_H)) = 1 - \theta g(p_H) - (1 - \theta) f(p_H), \\
(S'_t) \text{ w.p. } (1 - \theta) f(p_H).
\end{cases}$$
Where we used the following notations:

\[ f(p_H) = (1 - p_H p_E)^{\ln(2\theta)/p_H^2} \]

\[ g(p_H) = (1 - p_H^2)^{\ln(2\theta)/p_H^2} \]

**Lemma 5.** For \( p_H \) small enough, we get \( \mathbb{E}[S'_\infty] \leq \frac{2}{\ln(2\theta)/p_H^2} \).

And for \( p_H \) small enough, we get \( \mathbb{E}[S'_\infty] \leq \frac{2}{\ln(2\theta)/p_H^2} \).

**Lemma 6.** There exists a coupling of the random walks \( S_t \) and \( M_t \) as defined in (19) such that if \( M_t \geq \frac{\ln(2\theta)}{p_H^2} \) we get:

\[ M_{t+1} - M_t \leq S'_{t+1} - S'_t. \]

The proof of Lemma 6 is deferred to the Appendix D.6. Again the intuition is that if there are more than \( \frac{\ln(2\theta)}{p_H^2} \) agents in the system, then \( S_t \) grows faster than \( M_t \) because it only matches agents “up to” the first \( \frac{\ln(2\theta)}{p_H^2} \). We get that for every \( t \),

\[ M_t \leq \frac{\ln(2\theta)}{p_H^2} + S'_t. \tag{24} \]

Therefore:

\[
\mathbb{E}[N_{H,\infty}] \leq \frac{\ln(2\theta)}{p_H^2} + \mathbb{E}[S'_\infty] + 1
\leq \frac{\ln(2\theta) + 2/\ln(2\theta)}{p_H^2}.
\tag{25}
\]

where \( A_\theta = \ln(2\theta) + 2/\ln(2\theta) \).

**Remark.** We notice that similarly to the proof in the case (2), we derive a similar bound by considering \( n = \frac{\ln(2\theta)}{p_H^2} \) for \( 0 < \delta \). This leads to:

\[
\mathbb{E}[N_{H,\infty}] \leq \frac{\ln(2\theta/i-\delta)}{p_H^2} + \frac{(1 - \delta)/2}{\delta + o(p_H^2)}.
\]

In Section A.1 simulations show that \( \ln(2\theta)/(\theta p_H^2) \) seems to be a tight constant. However here again, proving tightness remains an open question.
D.5 Proof of Theorem 1 part 3

Proof. This proof follows the same procedure as the proof for $H$ agents. To study the $E$ agents, we study the Markov chain $T_t$ obtained with a simplified algorithm, that both disregards $H$ agents when looking for matches, and only considers the first $\frac{\ln(4)}{p_E}$ $E$ agents. Assuming that $T_t \geq \frac{\ln(4)}{p_E}$ and using the above notations, we get:

$$T_{t+1} = \begin{cases} 
(T_t + 1) \text{ with probability } (1 - \theta)(1 - p^2_E) \frac{\ln(4)}{p_E}, \\
(T_t - 1) \text{ when with probability } (1 - \theta)(1 - (1 - p^2_E) \frac{\ln(4)}{p_E}).
\end{cases}$$

Lemma 7. Using a coupling argument, we show that if $N_{E,t} \geq \frac{\ln(4)}{p_E}$ then $N_{E,t+1} - N_{E,t} \leq T_{t+1} - T_t$, and therefore $N_{E,t} \leq T_t + \frac{\ln(4)}{p_E}$.

Using the notation $x = \frac{(1 - p^2_E)\ln(4)/p_E}{1 - (1 - p^2_E)\ln(4)/p_E}$, the expected value of $T_\infty$ is given by:

$$\mathbb{E}[T_\infty] = \frac{x}{1 - x} = \frac{(1 - p^2_E)^{n/p_E}}{1 - 2(1 - p^2_E)^{n/p_E}} \leq \frac{e^{-\ln(4)}}{1 - 2e^{-\ln(4)}} = \frac{1}{2}. \tag{26}$$

Therefore taking $C_\theta = \ln(4)$. And this concludes the proof of Theorem 1.

D.6 Proofs of the technical lemmas

D.6.1 Proof of Lemma 1

We provide a formal proof that the random process $(N_{H,t}, N_{E,t})$ is indeed a positive recurrent Markov chain, and therefore has a steady-state distribution.

Lemma 1. For any $0 < p_H < 1$, $0 < p_L \leq 1$ and $0 < \theta \leq 1$, the number of agents in the pool under BilateralMatch $(N_{H,t}, N_{E,t})_{t\in\mathbb{N}}$ is a positive recurrent Markov chain and it converges to a steady-state distribution $(N_{H,\infty}, N_{E,\infty})$.

Proof. Because the algorithm is online and only allows for 2-ways, an incoming agent $v$ will either be matched immediately or join the pool. In the latter case, $v$ will never match in a 2-way to an agent arrived before $v$. Therefore, the system does not add any memory in terms of edge realizations between agents of the pool. Thus, keeping track of only the number of agents $(N_{H,t}, N_{E,t})_{t\in\mathbb{N}}$ is enough to characterize the system. For any time $t$, we consider the following events:
- \( A_H(t) \) is the event of an H agent arriving to the pool. We have \( A_H(t) = B(\theta) \).

- \( A_E(t) \) is the event of an E agent arriving, correlated with \( A_H(t) \) so that the two events are mutually exclusive. We have \( A_E(t) = 1 - A_H(t) \).

- \( H_H(t) \) is the event that an incoming H agent matches an H agent already present in the pool. We have \( H_H(t) = B(1 - (1 - p_H^2)^{N_H,t}) \).

- \( H_E(t) \) is the event that an incoming H matches an E. We have \( H_E(t) = (1 - H_H(t))B(1 - (1 - p_{HPH})^{N_E,t}) \).

- \( H_\emptyset(t) \) is the event that an incoming H doesn’t match immediately, and is therefore added to the pool. We have \( H_\emptyset(t) = 1 - H_H(t) - H_E(t) \).

- \( E_H(t) \) is the event that an incoming E matches an H. We have \( E_H(t) = B(1 - (1 - p_{EPH})^{N_H,t}) \).

- \( E_E(t) \) is the event that an incoming E matches an E. We have \( E_E(t) = (1 - E_H(t))B(1 - (1 - p_{E}^2)^{N_E,t}) \).

- \( E_\emptyset(t) \) the event that an incoming E doesn’t match immediately. We have \( E_\emptyset(t) = 1 - E_H(t) - E_E(t) \).

We obtain the following equations:

\[
\begin{align*}
N_{H,t+1} &= N_{H,t} + A_H(t)(H_\emptyset(t) - H_H(t)) - A_E(t)E_H(t), \\
N_{E,t+1} &= N_{E,t} - A_H(t)H_E(t) + A_E(t)(E_\emptyset(t) - E_E(t)),
\end{align*}
\]

(27)

In figure 8, we show the transitions for the markov chain that represents our dynamic system. We use the notations:

\[
\begin{align*}
a(n_H, n_E) &= \theta(1 - (1 - p_H^2)^{N_H,t}) + (1 - \theta)(1 - (1 - p_{EPH})^{N_H,t}) \\
b(n_H, n_E) &= \theta(1 - p_H^2)^{N_H,t}(1 - p_{EPH})^{N_{E,t}} \\
c(n_H, n_E) &= \theta(1 - p_H^2)^{N_H,t}(1 - (1 - p_{EPH})^{N_{E,t}}) + (1 - \theta)(1 - p_{EPH})^{N_{H,t}}(1 - (1 - p_{E}^2)^{N_{E,t}}) \\
d(n_H, n_E) &= (1 - \theta)(1 - p_{EPH})^{N_{H,t}}(1 - p_{E}^2)^{N_{E,t}}
\end{align*}
\]

(28)

Let \((n_H, n_E)\) be the state corresponding to a system with \( n_H \) (H) and \( n_E \) (E) agents. Let \( S_{\alpha,\beta} = \{(n_H, n_E)|n_H \leq \alpha, n_E \leq \beta\} \). Consider the Lyapunov function \( f : (n_H, n_E) \to n_H + n_E \). We observe that:

\[
\mathbb{E}[f(N_{H,t+1}, N_{E,t+1}) - f(N_{H,t}, N_{E,t})] = 2\theta(1 - p_H^2)^{N_{H,t}}(1 - p_{EPH})^{N_{E,t}} \\
+ (1 - \theta)(1 - p_{EPH})^{N_{H,t}}(1 - p_{E}^2)^{N_{E,t}} - 1,
\]

(29)
and that for large enough $\alpha$ and $\beta$, there exists $\gamma > 0$ such that
\[
\mathbb{E}[f(N_{H,t+1}, N_{E,t+1}) - f(N_{H,t}, N_{E,t}) | (N_{H,t}, N_{E,t}) \notin S_{\alpha,\beta}] \leq -\gamma.
\]

The Foster result allows us to conclude that $(N_H, N_E)$ is a positive recurrent Markov chain, and therefore it admits a unique steady-state distribution.

**D.6.2 Proof of Lemma 2**

Lemma 2 is composed of two parts. First we show in Claim 1 the existence of a steady-state distribution for $M_t$. Second we derive the existence of a suitable coupling.

**Lemma 2.** The Markov chain $M_t$ defined in (19) reaches a steady-state distribution $M_{\infty}$, and there exists a way to couple the Markov chains $(N_{H,t}, N_{H,t})$ and $M_t$ such that for every $t > 0$, $N_{H,t} \leq M_t + 1$. Therefore $\mathbb{E}[N_{H,\infty}] \leq \mathbb{E}[M_{\infty}] + 1$.

**Proof.** Recall that in Section D.2 we defined $M_t$ by:

\[
M_{t+1} = \begin{cases} 
(M_t + 1) & \text{with probability } \theta(1 - p_H^2)^{M_t}, \\
(M_t - 1) & \text{with probability } \theta(1 - (1 - p_H^2)^{M_t}) + (1 - \theta)(1 - (1 - p_{EPH})^{M_t}), \\
(M_t) & \text{with probability } (1 - \theta)(1 - p_{EPH})^{M_t}.
\end{cases}
\]
Claim 1. $M_t$ is a positive recurrent Markov chain. Therefore there exists a steady-state distribution $M_\infty$.

Let us now give a more precise definition in terms of random variables. We define:

- $A_H(t)$ and $A_E(t)$ are arrivals of $H$ and $E$ agents as defined as in section D.6.1.
- $\tilde{H}_H(t)$ is the event that the incoming $H$ agent matches with another $H$ in the "simplified system" $\tilde{H}_H(t) = B(1 - (1 - p_H^2)^{M_t})$.
- $\tilde{E}_H(t)$ is the event that the incoming $E$ agent matches with another $H$ in the "simplified system" $\tilde{E}_H(t) = B(1 - (1 - p_E p_H)^{M_t})$.

Note that these differ from $H_H(t)$ and $E_H(t)$ defined in section D.6.1 because they depend on $M_t$ instead of $N_{H,t}$. This leads us to an equation for $M_t$:

$$M_{t+1} = M_t + A_H(t)(1 - 2\tilde{H}_H(t)) - A_E(t)\tilde{E}_H(t). \quad (30)$$

Notice that this is very similar to (27), which gives us:

$$N_{H,t+1} = N_{H,t} + A_H(t)(1 - 2H_H(t) - H_E(t)) - A_E(t)E_H(t) \quad (31)$$

Now we define the coupling of the Markov chain $(N_{H,t}, N_{E,t})_{t \geq 0}$ with $(M_t)_{t \geq 0}$ we got from the simplified algorithm. We start with $M_t = 0$ and $N_{H,t} = N_{E,t} = 0$. We let the two chains evolve independently, except for times where $N_{H,t} = M_t$ or $N_{H,t} = M_t + 1$. In these cases, we correlate $H_H$ with $\tilde{H}_H$ in the following way:

- If $\tilde{H}_H = 1$ then $H_H = 1$, else $\tilde{H}_H = 0$ and $H_H = B(1 - (1 - p_H^2)^{N_{H,t}-M_t})$.
- If $\tilde{E}_H = 1$ then $E_H = 1$, else $\tilde{E}_H = 0$ and $E_H = B(1 - (1 - p_E p_H)^{N_{H,t}-M_t})$.

Notice that this does not modify the marginal distribution of $H_H$ ($E_H$) because when $N_{H,t} \geq M_t$, we have:

$$\mathbb{P}[\tilde{H}_H = 1] = 1 - (1 - p_H^2)^{M_t} \leq 1 - (1 - p_H^2)^{N_{H,t}} = \mathbb{P}[H_H = 1]$$

$$\mathbb{P}[\tilde{E}_H = 1] = 1 - (1 - p_E p_H)^{M_t} \leq 1 - (1 - p_E p_H)^{N_{H,t}} = \mathbb{P}[E_H = 1]$$

This coupling implies that with probability 1, we get for every realization of the random variables, $H_H \geq \tilde{H}_H$. 

35
We next prove by induction that for all $t \geq 0$, $N_{H,t} \leq M_t + 1$. Note that $M_0 = N_{H,0} = 0$. Further, if $N_{H,t} < M_t$ then $N_{H,t+1} \leq M_{t+1} + 1$. The only interesting case is when $N_{H,t} = M_t$ or $N_{H,t} = M_t + 1$. Then we obtain:

$$N_{H,t+1} - N_{H,t} = A_H(t)(1 - 2H(t) - H(t)) - A_E(t)E_H(t)$$

$$\leq A_H(t)(1 - 2H(t)) - A_E(t)E_H(t)$$

$$\leq A_H(t)(1 - 2H(t)) - A(t)E_H(t)$$

$$= M_{t+1} - M_t. \quad (32)$$

**Proof of Claim.** Let $S_\alpha = \{M_t | M_t \leq \alpha\}$ then for any $\epsilon > 0$ and for $\alpha$ large enough, we get for all $M_t > \alpha$ we have $(1 - \frac{2}{p_H})^M_t < \epsilon$ and $(1 - pE_H)^M_t < \epsilon$. Therefore

$$\mathbb{E}[M_{t+1} - M_t | M_t \notin S_\alpha] \leq \theta \epsilon - (1 - \epsilon) < -1/2.$$

**D.6.3 Proof of Lemma 3**

**Lemma 3.** $S_t$ reaches steady-state $S_\infty$ and we have

$$\mathbb{E}[S_\infty] = \frac{2\theta}{(1 - 2\theta) \ln \frac{1 - \theta}{1 - 2\theta} - pE_H} + o(1/pH). \quad (21)$$

**Proof.**

$$\theta - (1 - \theta)(1 - (1 - pE_H) \frac{n}{pE_H}) \leq \theta - (1 - \theta)(1 - e^{-n(\frac{pE_H}{2} + o(pE_H))})$$

$$= \theta - (1 - \theta) \left(1 - e^{-n(\frac{pE_H}{2} + o(pE_H))}\right). \quad (33)$$

Notice that the quantity we wish to bound is decreasing with $n$ and therefore it is only needed to find a bound for $n = \ln(\frac{1 - \theta}{1 - 2\theta})$. We have $e^{-n} = \frac{1 - 2\theta}{1 - \theta}$, therefore

$$\theta - (1 - \theta) \left(1 - e^{-n} \left(1 - \frac{npE_H}{2} + o(pE_H)\right)\right) = -(1 - 2\theta) \frac{npE_H}{2} + o(pE_H). \quad (34)$$

**D.6.4 Proof of Lemma 4**

Here we provide a rigorous derivation of the coupling for chains $M_t$ and $S_t$. 

36
Lemma 4. There exists a coupling of $M_t$ and $S_t$ such that for all $M_t > \frac{n}{p_{EPH}}$ we have:

$$M_t - M_{t-1} \leq S_t - S_{t-1}$$

Proof. In line with the notations introduced in in section [D.6.1] we define:

- $A_H(t)$ is the event of an H agent arriving to the pool.
- $A_E(t)$ is the event of an E agent arriving.
- $\bar{E}_H(t) = B((1 - (1 - p_{EPH})^{\frac{n}{p_{EPH}}})$ is the probability to match if there were exactly $\frac{n}{p_{EPH}}$ agents.

Let $S_t$ be the random walk defined by:

$$S_{t+1} = S_t + A_H(t) - A_E(t) \bar{E}_H(t). \quad (35)$$

We can verify that the transition states are what we expected:

$$S_{t+1} = \begin{cases} 
(S_t + 1) \text{ with probability } \theta, \\
(S_t - 1) \text{ with probability } (1 - \theta)(1 - (1 - p_{EPH})^{\frac{n}{p_{EPH}}}), \\
(S_t) \text{ with probability } (1 - \theta)(1 - p_{EPH})^{\frac{n}{p_{EPH}}}.
\end{cases}$$

Like previously, the main idea is to couple $M_t$ with $S_t$ such that when $M_t \geq \frac{n}{p_{EPH}}$ (i.e. when it is easy to match in chain $M_t$), every time there is a match for $S_t$, there is also a match for $M_t$. This is only possible because $P[\bar{H}_H = 1] \geq P[H_H = 1]$.

Formally, suppose that $M_t > \frac{n}{p_{EPH}}$. We modify the definition of $\bar{E}_H(t) = B(1 - (1 - p_{EPH})^{M_t})$ into $1 - (1 - \bar{E}_H(t)) \ast B((1 - p_{EPH})^{M_t - \frac{n}{p_{EPH}}})$ which does not change its distribution. This gives us:

$$M_{t+1} - M_t \leq A_H(t) - A_E(t) \bar{E}_H(t) = S_{t+1} - S_t.$$

\[\square\]

D.6.5 Proof of Lemma 5

Lemma 5. For $p_H$ small enough, we get $E[S_\infty] \leq \frac{2}{\ln(2\theta)p_H^2}$. 

Proof. Notice that:

$$f(p_H) = e^{-\ln(2\theta)p_H^2 + o(p_H)} = o(p_H^2). \quad (36)$$

$$g(p_H) = e^{-\ln(2\theta)(p_H^2 + p_H^4/2)} = \frac{1}{2\theta} \left(1 - \frac{\ln(2\theta)p_H^2}{2}\right) + o(p_H^2). \quad (37)$$
Note that $g(p_H) < 1$ for $p_H < 1$.

With $r = \frac{\theta g(p_H)}{1 - \theta g(p_H) + (1 - \theta)f(p_H)} < 1$ for $f(p_H)$ small enough, we get in steady-state $P[S'_\infty = i] = r^i(1 - r)$. Thus using Equations 39, 40 we get:

$$E[S'_\infty] = \frac{r}{1 - r}$$

$$= \frac{\theta g(p_H)}{1 - 2\theta g(p_H) + (1 - \theta)f(p_H)}$$

$$= \frac{(1/2 + o(1))}{(\ln(2\theta)p_H^2 + o(p_H^2)) + (1 - \theta)o(p_H^2)}$$

$$= \frac{1}{\ln(2\theta)p_H^2} + o(1/p_H^2).$$

(D.6.6) Proof of Lemma 6

Here we provide a rigorous derivation of the coupling for chains $M_t$ and $S'_t$.

Lemma 6. There exists a coupling of the random walks $S_t$ and $M_t$ as defined in (19) such that if $M_t \geq \frac{\ln(2\theta)}{p_H^2}$ we get:

$$M_{t+1} - M_t \leq S'_{t+1} - S'_t.$$

Proof. Recall that we defined:

$$f(p_H) := (1 - p_Hp_E)^{\ln(2\theta)/p_H^2}.$$  (39)

$$g(p_H) := (1 - p_H^2)^{\ln(2\theta)/p_H^2}.$$  (40)

Let:

- $Q(t) := B(1 - g(p_H))$.
- $P(t) := B(1 - f(p_H))$.

Let $S'_t$ be the random walk defined by:

$$S'_{t+1} = S'_t + A_H(t)(1 - 2Q(t)) - A_E(t)P(t),$$

It is easy to verify that this leads to the Markov transitions we expected:

$$S'_{t+1} = \begin{cases} 
(S'_t + 1) & \text{with probability } \theta g(p_H), \\
(S'_t - 1) & \text{w.p. } \theta(1 - g(p_H)) + (1 - \theta)(1 - f(p_H)) = 1 - \theta g(p_H) + (1 - \theta)f(p_H), \\
(S'_t) & \text{w.p. } (1 - \theta)f(p_H).
\end{cases}$$
We now wish to update the definition of $M_t$ to introduce some correlation with $S_t$ without modifying its marginal distribution. To do this we redefine $\hat{H}_H(t) := 1 - (1 - Q(t)) \ast B((1 - p_H^2)^M_t / g(p_H))$, and $\hat{E}_H(t) := 1 - (1 - P(t)) \ast B((1 - p_H p_E)^M_t / f(p_H))$ which does not change their marginal distributions.

Then if $M_t \geq \ln(2\theta) p_H^2$, we get:

$$M_{t+1} - M_t \leq A_H(t)(1 - Q(t)) - A_E(t)P(t) = S'_{t+1} - S'_t.$$ 

\[ \square \]

D.6.7 Proof of Lemma 7

**Lemma 7.** Using a coupling argument, we show that if $N_{E,t} \geq \ln(4) p_E^2$ then $N_{E,t+1} - N_{E,t} \leq T_{t+1} - T_t$, and therefore $N_{E,t} \leq T_t + \ln(4) p_E^2$.

**Proof.** Using notations introduced in section D.6.4, recall that $N_{E,t}$ follows:

\[
\begin{cases}
N_{H,t+1} = N_{H,t} + A_H(t)(H_0(t) - H_H(t)) - A_E(t)E_H(t), \\
N_{E,t+1} = N_{E,t} - A_H(t)H_E(t) + A_E(t)(E_0(t) - E_E(t)).
\end{cases}
\]

To study the $E$ agents, we study the Markov chain $T_t$ obtained with a simplified algorithm, that both disregards $H$ agents when looking for matches, and only considers the first $\frac{n}{p_E} E$ agents for $n$ suitably chosen. Assuming that $E_t \geq \frac{n}{p_E}$ and using the above notations, we get:

$$T_{t+1} = T_t + A_E(t)(1 - 2\hat{E}_E(t)),$$

Where $\hat{E}_E(t)$ is the indicator that the incoming $E$ agent is able to match to one of the first $\frac{n}{p_E} E$ agents in the pool.

We now introduce a correlated version of $E_E(t) = 1 - (1 - \hat{E}_E(t))B((1 - p_E^2)^{N_{E,t} - \frac{n}{p_E}}) \geq \hat{E}_E(t)$, which does not impact the marginal distribution of $E_E(t)$. We now have:

$$N_{E,t+1} - N_{E,t} = -A_H(t)H_E(t) + A_E(t)(1 - E_H(t) - 2E_E(t))$$

$$\leq A_E(t)(1 - 2E_E(t)) \leq A_E(t)(1 - 2\hat{E}_E(t)) = T_{t+1} - T_t.$$ 

And therefore $N_{E,t} \leq T_t + \frac{n}{p_E}$. 

\[ \square \]

E Chain matching: proof of Theorem 2

**Theorem 2.** Suppose that $p_E = 1$ and $0 < \theta \leq 1$. Under the ChainMatch policy, the market reaches steady-state and:

39
1. If \( \theta < 1 \), there exists a constant \( K_\theta \) such that:
\[
 w_H \leq \frac{K_\theta}{p_H}. \tag{4}
\]

2. If \( \theta = 1 \):
\[
 w_H \leq \frac{2 \ln(1/p_H)}{p_H}. \tag{5}
\]

The derivation of the proof involves several steps:

- First we define formally our system as a Markov chain, and we show that it is positive recurrent, and therefore reaches steady-state.

- We prove that the chain segments that are conducted in ChainMatch have known expected length (Proposition 2).

- Then we prove theorem 2. We separate the case \( \theta = 1 \) and the case \( \theta < 1 \).

- Finally we prove all the intermediate technical lemmas used.

### E.1 Markov Chain representation of ChainMatch

Because we consider only the case in which \( p_E = 1 \), there will never be any \( E \) agent in the system. Because there is no ambiguity, we will simplify the notations and write \( N_t \) instead of \( N_{H,t} \). Therefore we can count only the number of \( H \) agents in the pool. We consider the 1-dimensional Markov Chain \((N_t)_{t \geq 0}\) that represents the number of agents in the pool at time \( t \). From state \((N_t)\) all the states between 0 and \( N_t + 1 \) are reachable.

Note that the number of \( H \) agents only decreases when there arrives an agent (\( H \) or \( E \)) that is able to recieve from the bridge donor. This happens with probability \( \theta p_H + (1 - \theta) \) and a chain is then started. Suppose that there are \( n \) agents in the system, and we are currently running a chain. The probability that the bridge donor is able to match at least one of the agents is \( 1 - (1 - p_H)^n \). This allows us to compute the probability that a chain is of length \( k \leq N_t \) conditional on the fact that there were \( N_t \) agents just before the start of the chain.

\[
 N_{t+1} = \begin{cases} 
 (N_t + 1) \text{ with probability } \theta(1 - p_H), \\
 N_t - k \text{ with probability } (\theta p_H + (1 - \theta)) \times (1 - p_H)^{N_t-k} \times \prod_{i=0}^{k-1} (1 - (1 - p_H)^{N_t-i}), \forall k \leq N_t,
\end{cases}
\]

Note that the process \( N_t \) so defined has the Markov property.
E.2 Proof of Proposition 2

Proposition 2. Let \( L_\infty \) be the length of a new chain segment formed by ChainMatch policy in steady-state, and suppose \( p_E = 1 \). Then we have:

\[
\mathbb{E}[L_\infty] = \frac{\theta(1 - p_H)}{\theta p_H + (1 - \theta)} + 1.
\]

Note that \( L_t \) is only defined if a chain was started at time \( t \). The idea of the proof is that the expected number of \( H \) agents must stay constant. Therefore, the length of chain segments is inversely proportional with the frequency at which those chain segments are started.

Proof. In steady-state,

\[
0 = \mathbb{E}[N_t - N_{t+1}] = \mathbb{P}[N_{t+1} > N_t]\mathbb{E}[N_t - N_{t+1} | N_{t+1} > N_t] \\
+ \mathbb{P}[N_{t+1} \leq N_t]\mathbb{E}[N_t - N_{t+1} | N_{t+1} \leq N_t] \\
= \theta(1 - p_H)\mathbb{E}[N_t - N_{t+1} | N_{t+1} > N_t] \\
+ (\theta p_H + (1 - \theta))\mathbb{E}[(N_t - N_{t+1}) | N_{t+1} \leq N_t].
\]

(42)

Where we used the law of total expectation, the fact that \( \mathbb{P}[N_{t+1} > N_t] = \mathbb{P}[N_{t+1} = N_t + 1] = \theta(1 - p_H) \), the fact that \( \mathbb{P}[N_{t+1} \leq N_t] = \theta p_H + (1 - \theta) \). Lastly note that if \( N_{t+1} \leq N_t \) then a new chain segment has been formed with expected length of \( \mathbb{E}[L_t] = \mathbb{E}[(N_t - N_{t+1}) | N_{t+1} \leq N_t] \). Therefore:

\[
\theta(1 - p_H) = (\theta p_H + (1 - \theta))\mathbb{E}[L_t].
\]

\( \square \)

E.3 Proof of Theorem 2: case \( \theta = 1 \)

To prove Theorem 2 we will need the following technical claim (the proof is deferred to E.5.2).

Claim 2. For \( 0 \leq n \leq \infty \) let

\[
z_{n,i,p} = \prod_{j=0}^{i-1} (1 - (1 - p)^{n-j}).
\]

Then for any \( \alpha \in (0, 1) \) and \( \epsilon > 0 \) there exists \( p_0 \) such that for all \( p < p_0 \), for all \( n \geq \frac{2\ln(1/p_H)}{p_H} \)

\[
\sum_{i=0}^{n} z_{n,i,p} \geq n\alpha(1 - \frac{\epsilon}{1 - \epsilon}).
\]

41
As we shall see later, \( z_{n,i,p} \) represents the probability that a chain that is started when there are \( n \) agents in the pool reaches a length of more than \( i \). Therefore, \( \sum_{i=0}^{\infty} z_{n,i,p} \) is the expected length of such a chain for a given parameter \( p \). This lemma states that if there are “enough” agents in the pool (in particular at least a log factor more than \( \frac{1}{p} \)), then the chain length is of the order of the number of agents in the pool.

**Remark.** The constant 2 in \( \frac{2\ln(1/pH)}{pH} \) in claim 2 could be lowered to any \( 1 + \delta \) with \( \delta > 0 \) without any alterations to the proof. This is also true for the theorem 2.

**Proof of Theorem 2** The idea of the proof is to cut the expectation in two parts. Observe that

\[
\mathbb{E}[N_t] = \sum_{n=0}^{2\ln(1/pH)/pH} n\mathbb{P}[N_t = n] + \sum_{n=2\ln(1/pH)/pH}^{\infty} n\mathbb{P}[N_t = n].
\]

We bound the first part by

\[
\sum_{n=0}^{2\ln(1/pH)/pH} n\mathbb{P}[N_t = n] \leq \frac{2\ln(1/pH)}{pH}\mathbb{P}[N_t = n] \leq \frac{2\ln(1/pH)}{pH}.
\]

It is sufficient to show that there exists a constant \( C \) independent of \( p \) such that

\[
\sum_{n=2\ln(1/pH)/pH}^{\infty} n\mathbb{P}[N_t = n] \leq C.
\]

By the definition of the Chains removal algorithm we have that

\[
\mathbb{P}[L_t > x|N_t = n] = \mathbb{I}_{x \leq n} \prod_{i=0}^{x-1} (1 - (1 - p_H)^{n-i}) = z_{n,x,p_H}\mathbb{I}_{x \leq n}.
\]

Therefore

\[
\mathbb{E}[L_t] = \mathbb{E}_{N_t}[\mathbb{E}[L_t|N_t]]
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \mathbb{P}[L_t > i|N_t = n] \right) \mathbb{P}[N_t = n]
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} z_{n,i,p_H} \right) \mathbb{P}[N_t = n] \tag{43}
\]

\[
\geq \sum_{n=\frac{2\ln(1/pH)}{pH}}^{\infty} \left( \sum_{i=0}^{n} z_{n,i,p_H} \right) \mathbb{P}[N_t = n].
\]
By Claim 2, we obtain that for any $\alpha \in (0, 1)$ and $\epsilon \in (0, 1)$ there exists $p_0$ such that for all $p < p_0$:

$$
\mathbb{E}[L_t] \geq \sum_{n=\frac{2\ln(1/pH)}{pH}}^{\infty} \alpha \frac{\epsilon}{1-\epsilon} n \mathbb{P}[N_t = n].
$$

Therefore

$$
\frac{\mathbb{E}[L_t]}{\alpha(1-\epsilon)} \geq \sum_{n=\frac{2\ln(1/pH)}{pH}}^{\infty} n \mathbb{P}[N_t = n].
$$

We prove in Lemma 8 (cf appendix E.5.1) that the algorithm leads to a market with steady-state behavior, therefore we can apply Proposition 2 and:

$$
\mathbb{E}[L_t] = \frac{\theta(1-p_H)}{\theta p_H + (1-\theta)}
$$

This concludes our proof.

$$
\mathbb{E}[N_t] = \sum_{n=0}^{\frac{2\ln(1/pH)}{pH}} n \mathbb{P}[N_t = n] + \sum_{n=\frac{2\ln(1/pH)}{pH}}^{\infty} n \mathbb{P}[N_t = n] 
\leq \frac{2\ln(1/pH)}{pH} + \mathbb{E}[L_t] \frac{\alpha}{\alpha(1-\epsilon)}
\leq \frac{2\ln(1/pH)}{pH} + \frac{1}{pH \alpha(1-\epsilon)}. \tag{44}
$$

**Remark.** Note that this proof also applies to the case where we have $\theta < 1$:

$$
\mathbb{E}[L_t] = \frac{\theta(1-p_H)}{\theta p_H + (1-\theta)} \leq \frac{\theta}{(1-\theta)}.
$$

And by taking $\alpha = 1/2$ and $\epsilon = 1/4$ we get:

$$
\mathbb{E}[N_t] \leq \frac{2\ln(1/pH)}{pH} + \frac{3\theta}{(1-\theta)}.
$$

However, this bound is weaker than the bound that we prove in the special case $\theta < 1$. Put simply, this stems from the fact that the cut of the expectation in two parts is not optimal.

### E.4 Proof of Theorem 2: case $\theta < 1$

**Proof.** This proof follows the outline of the proof for the case $\theta = 1$. Here we change where we cut the expectation in two parts. Let $c$ be function of $p_H$ to be determined later. We get:

$$
\mathbb{E}[N_t] = \sum_{n=0}^{\frac{c}{pH}} n \mathbb{P}[N_t = n] + \sum_{n=\frac{c}{pH}}^{\infty} n \mathbb{P}[N_t = n].
$$
Similarly to the previous proof, we have

\[ \sum_{n=0}^{\phi_{PH}} nP[N_t = n] \leq \frac{c}{PH}. \]

And

\[ \tilde{l} = E[E[L_t | N_t]] \]
\[ \geq \sum_{n=PH}^{\infty} \left( \sum_{i=0}^{n} z_{n,i,PH} \right) P[N_t = n]. \]

(45)

We now use another technical claim:

Claim 3. There exists \( k < 1/2 \) such that for any \( n > 0 \), there exists \( p_0 > 0 \) such that for all \( p_H < p_0 \), for all \( n \geq \frac{2}{(1-\phi_H)PH} \):

\[ \sum_{i=0}^{n} z_{n,i,PH} \geq nP_H(1 - \frac{k}{1-k}) \]

Therefore, choosing \( c \) = \( \frac{2}{1-\phi_H} \) we get

\[ \sum_{n=\frac{2}{(1-\phi_H)PH}}^{\infty} nP[N_t = n] \leq \frac{\tilde{l}}{P_H(1 - \frac{k}{1-k})}. \]

Given that the market reaches steady-state and \( \theta < 1 \) we obtain that

\[ \tilde{l} = l^* = \frac{\theta(1-P_H)}{\theta P_H + (1-\theta)}. \]

Finally, we get that

\[ E[N_t] \leq \frac{2}{(1-P_H)PH} + \frac{\theta(1-P_H)}{(\theta P_H + 1-\theta)(P_H(1 - \frac{k}{1-k}))} = K_\theta \frac{P_H}{PH} + o(\frac{1}{PH}) \]

Where \( C_\theta = 2 + \frac{\theta}{(1-\theta)\frac{1-k}{1-2k}} \).

\[ \Box \]

Remark. Notice that the assumption that \( \theta < 1 \) is crucial in order for this bound to be tight, otherwise, it goes to the order of \( \frac{1}{PH} \). This means that this bound gets very bad as \( \theta \) gets closer to 1, and in practice it is observed that this is not tight when \( \theta \geq 0.7 \). A search for a tighter bound could be achieved through a splitting of the expectation in a manner that depends on \( \theta \). The exact value of the tightest bound remains an open question.
E.5 Proof of technical claims

E.5.1 Existence of steady-state

**Lemma 8.** For any arrival rate $0 \leq \theta \leq 1$ of $H$ agents and $p$ sufficiently small, the chain algorithm leads to steady-state.

**Proof.** The system has the Markov Property (it is memoryless by construction), and its transitions are given by:

$$
\begin{cases}
    N_{t+1} = N_t + 1 \text{ w.p. } \theta(1 - p_H) \\
    \forall k \in [0; N_t], N_{t+1} = (N_t - k) \text{ w.p. } (\theta p_H + (1 - \theta)) \sum_{i=0}^{n} z_{n,i,p} \\
    (\theta p_H + (1 - \theta))^N \prod_{i=0}^{N_t-i} (1 - (1 - p_H)^{N_t-i})
\end{cases}
$$

The first item corresponds to the case where an $H$ agent arrives ($\theta$) and isn’t able to receive from the bridge donor ($1 - p$). The second corresponds to the case where the incoming agent (either $E$ with probability $(1 - \theta)$ or $H$ with probability $\theta p_H$) is able to receive from the bridge donor. In that case we consider that a chain is started (of initial length 0). Then for any $k$ between 0 and $N_t$, $\prod_{i=0}^{N_t-i} (1 - (1 - p_H)^{N_t-i})$ is the probability that the chain is of length at least $k$. $(1 - p_H)^{N_t-k}$ corresponds to the fact that the chain isn’t able to continue beyond $k$.

We wish to show that there exists $\tilde{n}$ and $\gamma > 0$ such that

$$
\mathbb{E}[N_{t+1} - N_t | N_t > \tilde{n}] < -\gamma
$$

$$
\mathbb{E}[N_{t+1} - N_t | N_t > \tilde{n}] \mathbb{P}[N_t > \tilde{n}] = \sum_{n=\tilde{n}}^{\infty} \left( \theta(1 - p_H) - (\theta p_H + (1 - \theta)) \sum_{i=0}^{n} z_{n,i,p} \right) \mathbb{P}[N_t = n]
$$

(46)

As a reminder, $z_{n,i,p} = \prod_{j=0}^{i-1} (1 - (1 - p)^{n-j})$ for $0 \leq n \leq \infty$, and we have the property that for all $n \geq m$, $z_{n,i,p} \geq z_{m,i,p}$.

Let us fix $\alpha = 1/2, \epsilon = 1/4$. From the technical lemma there exists $p_0$ such that if $p < p_0$ and $n > \frac{2\ln(1/p)}{p}$ then we get:

$$
\sum_{i=0}^{n} z_{n,i,p} \geq \frac{n}{3}
$$

Let us consider $\tilde{n} = \max \left( \frac{2\ln(1/p)}{p}, \frac{3(\theta(1-p)+\gamma)}{(1-\theta)+\theta p} \right)$. For $n \geq \tilde{n}$:

$$
\sum_{i=0}^{n} z_{n,i,p} \geq n/4 \geq \frac{(\theta(1-p)+\gamma)}{(1-\theta)+\theta p}
$$
\[ \theta(1 - pH) - (\theta pH + (1 - \theta)) \sum_{i=0}^{n} z_{n,i,p} \leq -\gamma \]

Therefore there exists \( p_0 \) such that for \( p < p_0 \),

\[ \mathbb{E}[N_{t+1} - N_t | N_t > \frac{2\ln(1/p)}{p}] \mathbb{P}[N_t > \frac{2\ln(1/p)}{p}] \leq -\gamma \mathbb{P}[N_t \geq \frac{2\ln(1/p)}{p}] \]

\[ \mathbb{E}[N_{t+1} - N_t | N_t > \frac{2\ln(1/p)}{p}] \leq -\gamma \]

Therefore \( N_t \) is a recurring Markov Chain, and it has a steady-state distribution. \( \square \)

### E.5.2 Proof of Claim 2

**Claim 2.** For \( 0 \leq n \leq \infty \) let

\[ z_{n,i,p} = \prod_{j=0}^{i-1} (1 - (1 - p)^{n-j}). \]

Then for any \( \alpha \in (0, 1) \) and \( \epsilon > 0 \) there exists \( p_0 \) such that for all \( p < p_0 \), for all \( n \geq \frac{2\ln(1/pH)}{pH} \)

\[ \sum_{i=0}^{n} z_{n,i,p} \geq n\alpha (1 - \frac{\epsilon}{1 - \epsilon}). \]

**Proof.** Let us consider any \( 0 < \alpha < 1 \) independent of \( n \) and \( p \). In order to be able to get a lower bound of the term inside the product, we only consider the truncated sum:

\[ \sum_{i=0}^{n} z_{n,i,p} \geq \prod_{j=0}^{i-1} (1 - (1 - p)^{n-j}) \]

Using the fact that for all \( j \leq \alpha n \), we get \( (1 - (1 - p)^{n-j}) \geq (1 - (1 - p)^{n(1-\alpha)}) \), we can simplify:

\[ \sum_{i=0}^{n} z_{n,i,p} \geq \sum_{i=0}^{n\alpha} (1 - (1 - p)^{n(1-\alpha)})^i \]

This is a geometric sum with parameter \( y_n = 1 - (1 - p)^{n(1-\alpha)} \) therefore we get

\[ \sum_{i=0}^{n} z_{n,i,p} \geq \frac{1 - y_n^{n\alpha+1}}{1 - y_n} \geq \frac{1 - y_n^{n\alpha}}{1 - y_n} \]

(47)

Let us now expand \( y_n^{\alpha n} \):

\[ y_n^{\alpha n} = (1 - (1 - p)^{n(1-\alpha)})^{\alpha n} = 1 - n\alpha(1 - p)^{n(1-\alpha)} + \sum_{k=2}^{n\alpha} \binom{\alpha n}{k} (-1)^{k} (1 - p)^{n(1-\alpha)}^k. \]

(48)
And
\[
1 - y_n^{\alpha n} \geq n\alpha (1 - p)^{n(1 - \alpha)} - \sum_{k=2}^{n\alpha} \binom{\alpha n}{k} (1 - p)^{kn(1 - \alpha)}
\] (49)

**Claim 4.** For all $\epsilon > 0$ there exists $p_0$ such that for all $p < p_0$, for all $n \geq \frac{2\ln(1/p)}{p}$,
\[
(1 - p)^{n(1 - \alpha)} \leq \frac{\epsilon}{n\alpha}
\]

Using Claim 4 together with the fact that $\binom{\alpha n}{k} \leq (\alpha n)^k$ we get that for $p$ small enough and $n \geq \frac{2\ln(1/p)}{p}$,
\[
\sum_{k=2}^{n\alpha} \binom{\alpha n}{k} (1 - p)^{kn(1 - \alpha)} \leq \sum_{k=2}^{n\alpha} (\alpha n(1 - p)^{n(1 - \alpha)})^k
\]
\[
= (\alpha n(1 - p)^{n(1 - \alpha)})^2 \sum_{k=0}^{n\alpha - 2} (\alpha n(1 - p)^{n(1 - \alpha)})^k
\]
\[
\leq (\alpha n(1 - p)^{n(1 - \alpha)})\epsilon \sum_{k=0}^{n\alpha - 2} \epsilon^k
\]
\[
\leq (\alpha n(1 - p)^{n(1 - \alpha)})\frac{\epsilon}{1 - \epsilon}.
\] (50)

Putting together (49) and (50), this proves the claim:
\[
\sum_{i=0}^{n} z_{n,i,p} \geq \frac{1 - y_n^{\alpha n}}{1 - y_n} \geq \frac{n\alpha(1 - p)^{n(1 - \alpha)}(1 - \frac{\epsilon}{1 - \epsilon})}{(1 - p)^{n(1 - \alpha)}}
\]
\[
= n\alpha(1 - \frac{\epsilon}{1 - \epsilon}).
\] (51)

**Proof of Claim 4.** Let us pose $\tilde{n}_p = \frac{2\ln(1/p)}{p}$. We notice that the function $\phi : n \mapsto n\alpha(1 - p)^{n(1 - \alpha)}$ is decreasing after $n^* = \frac{1}{(1 - \alpha)\ln(1/(1 - p))}$. We have $\frac{1}{(1 - \alpha)\ln(1/(1 - p))} \leq \frac{1}{(1 - \alpha)p}$ and there-
fore for \( p \) small enough \( \tilde{n}_p \geq \frac{1}{(1-\alpha) \ln(1-p)} \) and for \( n \geq \tilde{n}_p \) we have \( \phi(n) \leq \phi(\tilde{n}_p) \):

\[
n\alpha(1-p)^{n(1-\alpha)} \leq \tilde{n}_p \alpha(1-p)^{\tilde{n}_p(1-\alpha)} \\
 \leq \frac{2 \ln(1/p)}{\alpha} (1-p)^{2 \ln(1/p)(1-\alpha)} \\
 = \frac{2 \ln(1/p)}{\alpha} e^{2 \ln(1/p)(1-\alpha) \ln(1-p)} \\
 \leq \frac{2 \ln(1/p)}{\alpha} e^{-2 \ln(1/p)(1-\alpha)} \\
 = \frac{2 \ln(1/p)}{\alpha} \alpha^2 e^{-\alpha} \\
 \rightarrow 0.
\]

Therefore for all \( \epsilon > 0 \) there exists \( p_0 \) such that for all \( p < p_0 \), for all \( n \geq \frac{2 \ln(1/p)}{\alpha} \),

\[
(1-p)^{n(1-\alpha)} \leq \frac{\epsilon}{n\alpha}
\]

\( \square \)

E.5.3 Proof of Claim 3

The proof of Claim 3 follows the same ideas that were used in proving Claim 2. Notice that we only ask \( n \) to grow linearly in \( \frac{1}{p} \). This is at the expense of a slightly weaker statement (there exists \( k \) instead of for all \( \epsilon \)).

Claim 3. There exists \( k < 1/2 \) such that for any \( n > 0 \), there exists \( p_0 > 0 \) such that for all \( p_H < p_0 \), for all \( n \geq \frac{2}{(1-p_H)p_H} \):

\[
\sum_{i=0}^{n} z_{n,i,p_H} \geq np_H (1 - \frac{k}{1-k})
\]

Proof. As a reminder, for \( 0 \leq n \leq \infty \), we have \( z_{n,i,p_H} = \prod_{j=0}^{i-1} (1 - (1 - p_H)^{n-j}) \).

Let us consider any \( p_0 > 0 \), and \( p_H < p_0 \). Using the fact that \( (1 - (1 - p_H)^{n-j}) \geq (1 - (1 - p_H)^{n(1-p_H)}) \) for all \( j \leq p_H n \), we can truncate the sum:

\[
\sum_{i=0}^{n} z_{n,i,p_H} \geq \sum_{i=0}^{p_H} \prod_{j=0}^{i-1} (1 - (1 - p_H)^{n-j}) \geq \sum_{i=0}^{p_H} (1 - (1 - p_H)^{n(1-p_H)})^i
\]

Writing \( y_n = 1 - (1 - p_H)^{n(1-p_H)} \) we get

\[
\sum_{i=0}^{n} z_{n,i,p_H} \geq \frac{1 - y_n^{p_H+1}}{1 - y_n} \geq \frac{1 - y_n^{p_H}}{1 - y_n}
\]

(53)
Claim 5. For $n \geq \tilde{n}_p$, there exists $k < 1/2$ such that:

$$np_H (1 - p_H)^n (1 - p_H) \leq k$$

This yields

$$y_n^{pH} = \left(1 - (1 - p_H)^n (1 - p_H)\right)^{pH}$$

$$= 1 - np_H (1 - p_H)^n (1 - p_H) + \sum_{i=2}^{np_H} \left(\begin{array}{c} p_H n \\ i \end{array}\right) (-1)^i (1 - p_H)^n (1 - p_H)$$

(54)

And

$$1 - y_n^{pH} \geq np_H (1 - p_H)^n (1 - p_H) - \sum_{i=2}^{np_H} \left(\begin{array}{c} p_H n \\ i \end{array}\right) (1 - p_H)^n (1 - p_H)$$

(55)

Using the fact that $\left(\begin{array}{c} p_H n \\ i \end{array}\right) \leq (p_H n)^i$ we get that

$$\sum_{i=2}^{np_H} \left(\begin{array}{c} p_H n \\ i \end{array}\right) (1 - p_H)^n (1 - p_H) \leq \sum_{i=2}^{np_H} (p_H n (1 - p_H)^n (1 - p_H))^i$$

$$= (p_H n (1 - p_H)^n (1 - p_H))^2 \sum_{i=0}^{np_H-2} (p_H n (1 - p_H)^n (1 - p_H))^i$$

$$\leq (p_H n (1 - p_H)^n (1 - p_H))^{np_H-2} \sum_{i=0}^{k} k^i$$

$$\leq \frac{k}{1 - k}$$

(56)

Therefore, putting together (55) and (56), we get:

$$\sum_{i=0}^{n} z_{n,i,p_H} \geq \frac{1 - y_n^{pH}}{1 - y_n}$$

$$\geq np_H (1 - p_H)^n (1 - p_H) (1 - \frac{k}{1 - k})$$

$$\geq np_H (1 - \frac{k}{1 - k}).$$

(57)

Proof. Let us pose $\tilde{n}_p = \frac{c}{p_H}$. We notice that $np_H (1 - p_H)^n (1 - p_H)$ is maximal for $n = \frac{1}{(1 - p_H) \ln(1 - p_H)}$ and is decreasing for larger values of $n$. We have $\frac{1}{(1 - p_H) \ln(1 - p_H)} \leq \frac{1}{(1 - p_H) p_H}$ and therefore if $c > \frac{1}{1 - p_H}$ then for $p_H$ small enough $\tilde{n}_p \geq \frac{1}{(1 - p_H) p_H}$.\footnote{In everything that follows we consider $c = \frac{2}{1 - p_H}$, but any constant $1 + \delta$ could work instead of 2.} We also fix $\tilde{n}_p = \frac{2}{(1 - p_H) p_H}$.\footnote{We fix $\tilde{n}_p = \frac{2}{(1 - p_H) p_H}$ to ensure the inequality holds for small $p_H$.}
For \( n \geq \tilde{n}_p \) we have

\[
np_H(1 - p_H)^{n(1-p_H)} \leq \tilde{n}_p p_H(1 - p_H)\tilde{n}_p(1-p_H)
\]

\[
\leq \frac{2}{(1 - p_H)p_H}p_H(1 - p_H)^{(1-p_H)p_H(1-p_H)}
\]

\[
\leq \frac{2}{(1 - p_H)}e^{\frac{1}{p_H} \ln(1-p_H)}
\]

\[
\leq \frac{2}{(1 - p_H)}e^{-2+o(1)}
\]

\[
\leq 2e^{-2} + o(1).
\]

Using the fact that \( 2e^{-2} < \frac{1}{2} \) we get that there exists \( k < 1/2 \), and \( p_0 > 0 \) such that

\[
\frac{2e^{2\ln(1-p_H)/p_H}}{1-p_H} < k \text{ for all } p_H < p_0. \]

Therefore for \( n \geq \tilde{n}_p \), we have:

\[
np_H(1 - p_H)^{n(1-p_H)} \leq k.
\]

\[
\Box
\]

F Proof of Proposition

We now modify the \textit{BilateralMatch(H)} algorithm into \textit{BilateralMatch(E)} which gives priority to \( E \) nodes instead of \( H \) nodes. We show that when \( p_E = \frac{1}{41} \), Theorem 1 still holds. Namely:

\textbf{Theorem 3.} Under \textit{BilateralMatch(E)}, the market reaches steady-state. Furthermore, there exist positive constants \( A'_\theta \), \( B'_\theta \), and \( C'_\theta \) such that:

- If \( \theta < 1/2 \), then

\[
w_H \leq \frac{A'_\theta}{p_E p_H}.
\]

(59)

- If \( \theta \geq 1/2 \), then

\[
w_H \leq \frac{B'_\theta}{p_H^2}.
\]

(60)

- For \( 0 \leq \theta < 1 \)

\[
w_E \leq \frac{C'_\theta}{p_E^2}.
\]

(61)

\footnote{From the simulations we conducted, it seems that this result extends to the case where \( p_E < 1 \). However due to an increased dimensionality of the problem, we haven’t been able to prove the Theorem in this general case.}
The proof of Theorem 3 is organised as follows:

First in Subsection F.1 we provide a Markov chain representation of the dynamic system under $BilateralMatch(E)$. Next, in Subsection F.2 we couple the chain with a simplified Markov chain for which the computation of the steady-state equilibrium is tractable. In Subsection F.3 we analyze the simplified chain, and show that bounding its eigenvalues is sufficient to provide a bound on the steady-state expected number of agents.

F.1 Markov chain representation

Because we assume $p_E = 1$, the $E$ agents match to each other with probability 1, there can never be more than 1 $E$ agent in the system at any given time. Therefore the system can be represented by a Markov chain $(N_{H,t},N_{E,t})$ where $N_{H,t} \geq 0$ and $n_{E,t} \in \{0,1\}$. Let $\pi_{n_{H},n_{E}}$ be the steady-state probability of the chain being in state $(n_H,n_E)$. We have:

$$
\mathbb{E}[N_{H,\infty}] = \sum_{n_H=0}^{\infty} n_H (\pi_{n_H,0} + \pi_{n_H,1}).
$$

Because we removed the $n_E$ dimension of the problem, and in order to simplify the notations, we will use $n$ instead of $n_H$ in the equations that follow.

In figure 9, we show the transitions for the markov chain that represents our dynamic system.

Figure 9: Transitions Markov chain resulting from $Bilateral(E)$ in the case $p_E = 1$. In figure 9, we show the transitions for the markov chain that represents our dynamic system.
\begin{equation}
\begin{aligned}
a(n, 0) &= \theta(1 - (1 - p_H^2)^n) + (1 - \theta)(1 - (1 - p_H)^n) \\
b(n, 0) &= \theta(1 - p_H^2)^n \\
d(n, 0) &= (1 - \theta)(1 - p_H)^n \\
a(n, 1) &= \theta(1 - p_H)(1 - (1 - p_H^2)^n) \\
b(n, 1) &= \theta(1 - p_H)(1 - p_H^2)^n \\
c(n, 1) &= \theta(1 - (1 - p_H)) + (1 - \theta)
\end{aligned}
\end{equation}

$a(n, 0)$ corresponds to the case where an \(E\) or an \(H\) agent arrives and matches an existing \(H\), $a(n, 1)$ corresponds to the event where an \(H\) arrives to the system, doesn’t match the existing \(E\) (to which we give priority), and matches one of the existing \(H\) agents. Similarly, \(b(n, 0)\) corresponds to the case where an \(H\) agent arrives to the system and is not able to match to the existing \(H\), and \(b(n, 1)\) corresponds to the case where the incoming \(H\) is not able to match to either the \(E\) or another \(H\). Finally, \(d(n, 0)\) corresponds to the case where an incoming \(E\) is unable to match any of the existing \(H\) agents, and \(c(n, 1)\) corresponds to the event where either an \(H\) or an incoming \(E\) agent matches the existing \(E\).

### F.2 Coupling to a simplified Markov Chain

We now wish to simplify the Markov chain in order to provide a tractable computation of the steady-state distribution. To do this, we notice that it suffice to upper bound the “forward transitions” of the upper states (corresponding to 1 \(E\) agent) using $b'(n, 1) = \theta(1 - p_H^2)^n \geq b(n, 1)$ and to lower bound the “backwards transitions” of the upper states using $a'(n, 1) = 0 \leq a(n, 1)$. Note that this new Markov Chain does not satisfy the conservation of flow, therefore we add self loops (to the upper states) (not shown in Figure 10) so that the outgoing probabilities out of every node sum up to 1. This yields the chain \((N'_{H,t}, N'_{E,t})\) for which we plot the transitions in Figure 10.

Using similar coupling ideas we develop in Appendix D and E, we can prove that for all $t$, \(N_{H,t} \leq N'_{H,t} + 1\).

### F.3 Analysing the simplified Markov Chain

Flow conservation for the coupled chain between $n$ and $n + 1$ yields:

\begin{equation}
\begin{aligned}
b(n, 0)\pi_{n,0} + b'(n, 1)\pi_{n,1} &= a(n + 1, 0)\pi_{n+1,0} \\
(\pi_{n,0} + \pi_{n,1})\theta(1 - p_H^2)^n &= \pi_{n+1,0} \left(1 - \theta(1 - p_H^2)^{n+1} - (1 - \theta)(1 - p_H)^{n+1}\right)
\end{aligned}
\end{equation}
Flow conservation for the coupled chain out of node \((n, 1)\) yields:

\[
(b'(n, 1) + c(n, 1))\pi_{n,1} = b'(n - 1, 1)\pi_{n-1,1} + d(n, 0)\pi_{n,0}
\]

\[
\pi_{n,1} (\theta(1 - p_H^2)^n + \theta p_H + (1 - \theta)) = \theta(1 - p_H^2)^{n-1}\pi_{n-1,1} + (1 - \theta)(1 - p_H^n)\pi_{n,0}
\]

To simplify notations, we introduce

\[
u_n = \frac{(1 - \theta)(1 - p_H^n)}{\theta(1 - p_H^2)^n + (\theta p_H + 1 - \theta)},
\]

\[
v_n = \frac{(1 - \theta)(1 - p_H^n)}{\theta(1 - p_H^2)^n + (\theta p_H + 1 - \theta)}.
\]

Equation (64) simplifies into:

\[
\pi_{n,1} = u_n\pi_{n-1,1} + v_n\pi_{n,0},
\]

and (63) simplifies into:

\[
\pi_{n+1,0} = g_{n+1}\theta(1 - p_H^n)(\pi_{n,0} + \pi_{n,1})
= g_{n+1}\theta(1 - p_H^n)(1 + v_n)\pi_{n,0} + u_n\pi_{n-1,1}).
\]
Putting everything together, we get:

$$
\begin{pmatrix}
\pi_{n,1} \\
\pi_{n,0} \\
\pi_{n+1,0}
\end{pmatrix} =
\begin{pmatrix}
u_n & 0 & v_n \\
0 & 0 & 1 \\
g_{n+1,0} (1 - p_H^2) u_n & 0 & g_{n+1,0} (1 - p_H^2) (1 + v_n)
\end{pmatrix}
\begin{pmatrix}
\pi_{n-1,1} \\
\pi_{n-1,0} \\
\pi_{n,0}
\end{pmatrix}
\tag{69}
$$

We call this matrix $A_n$. In Section F.3.1 we show that for $n$ large enough, the eigenvalues of $A_n$ are bounded by a constant strictly less than 1. Using a Lyapunov argument, this justifies that the system reaches steady-state. In Section F.3.2 we use our bound on the eigenvalues of the matrix to upper-bound $\mathbb{E}[N'_{H,\infty}]$. This gives an upper-bound on $\mathbb{E}[N_{H,\infty}]$ which completes the proof.

F.3.1 Bounding the matrix eigenvalues

Let $\lambda_1^{(n)}, \lambda_2^{(n)}$ and $\lambda_3^{(n)}$ be its eigenvalues, and let $\lambda_n = \max(|\lambda_1^{(n)}|, |\lambda_2^{(n)}|, |\lambda_3^{(n)}|)$. We wish to prove that there exists $\lambda < 1$ such that for all $n$, $\lambda_n < \lambda$. To do this, we will look separately into the cases $\theta > 1/2$ and $\theta < 1/2$.

**Case $\theta > 1/2$ :** Let $k_\epsilon$ be a constant (We allow this $k_\epsilon$ to depend on $\theta$, but it is independent of $p_H$ and $n$) to be fixed later. Let us consider $n^* = k_\epsilon / p_H$. For any $n \geq n^*$ we have $u_n \leq \frac{\theta e^{-k_\epsilon}}{\theta p_H + 1 - \theta}$ and $v_n \leq \frac{(1 - \theta) e^{-k_\epsilon} / p_H}{\theta p_H + 1 - \theta} = o(p_H)$. Therefore for any $\epsilon > 0$ there exists a $k_\epsilon$ large enough such that $u_n \leq \epsilon$, $v_n \leq \epsilon$, $g_{n+1} \theta (1 - p_H^2) u_n \leq \epsilon$ and $g_{n+1} \theta (1 - p_H^2) (1 + v_n) \leq \epsilon$.

$$
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

All the eigenvalues of $A$ are 0, therefore there exists a small perturbation $\epsilon$ such that for $n \geq n^* = k_\epsilon / p_H^2$, we get that $\lambda_i < 1/2$.

**Case $\theta < 1/2$ :** Let us consider $n^* = k_\epsilon / p_H$. For any $n \geq n^*$ we have $(1 - p_H^2)^n = 1 + o(1)$

$$
u_n = \frac{\theta}{\theta p_H + 1 - \theta} + o(\theta) = \frac{\theta}{1 - \theta (1 - \theta) e^{-k_\epsilon}} + o(\theta), \quad g_n = \frac{1}{1 - \theta (1 - \theta) e^{-k_\epsilon}} \quad \text{and} \quad v_n \leq \frac{(1 - \theta) e^{-k_\epsilon'}}{\theta p_H + 1 - \theta} = \frac{(1 - \theta) e^{-k_\epsilon}}{1 + \theta p_H}
$$

Furthermore:

$$
g_{n+1} \theta (1 - p_H^2)^n u_n = \frac{\theta^2}{(1 - \theta - (1 - \theta) e^{-k_\epsilon})(1 + \theta p_H)} + o(1)
$$

and

$$
g_{n+1} \theta (1 - p_H^2)^n (1 + v_n) = \frac{\theta (1 + v_n)}{(1 - \theta - (1 - \theta) e^{-k_\epsilon})}. $$

Therefore for any $\epsilon$ there exists a $k_\epsilon'$ large enough such that:

$$v_n \leq \epsilon$$

54
\[ g_{n+1} \theta (1 - p_H^2)^n u_n \leq \frac{\theta^2}{(1 - \theta)(1 + \theta p_H)} + \epsilon. \]

\[ g_{n+1} \theta (1 - p_H^2)^n (1 + v_n) \leq \frac{\theta}{(1 - \theta)} + \epsilon. \]

Let \( B \) be the matrix defined by:

\[
B = \begin{pmatrix}
\frac{\theta}{1 + \theta p_H} & 0 & 0 \\
0 & 0 & 1 \\
\frac{\theta^2}{(1 - \theta)(1 + \theta p_H)} & 0 & \frac{\theta}{(1 - \theta)}
\end{pmatrix}
\]

Its eigenvalues are \( \frac{\theta}{1 - \theta}, \frac{\theta}{1 + \theta p_H} \) and 0, which are all strictly less than 1. By the same perturbation argument as before, there exists an \( \epsilon \) small enough such all the eigenvalues of \( A_n \) are also strictly less than \( \lambda = \left( \max\left( \frac{\theta}{1 - \theta}, \frac{\theta}{1 + \theta p_H} \right) + 1 \right) / 2 < 1 \).

Therefore for both cases, there exists a \( \lambda \) that depends only on \( \theta \) and not \( p_H \), an \( \epsilon \) such that for \( n^* = k' \) and for all \( n > n^* \), the maximum eigenvalue \( \lambda_n \) of \( A_n \) is such that \( |\lambda_n| \leq |\lambda| < 1 \)

### F.3.2 Bounding the expected waiting time for \( H \) agents.

Let us consider \( x_n = ||(\pi_{n,1}, \pi_{n,0}, \pi_{n+1,0})||^2 = \pi_{n,1}^2 + \pi_{n,0}^2 + \pi_{n+1,0}^2 \). We have

\[
(\pi_{n,0} + \pi_{n,1}) = \sqrt{\pi_{n,0}^2} + \sqrt{\pi_{n,1}^2} \leq \sqrt{\pi_{n,1}^2 + \pi_{n,0}^2 + \pi_{n+1,0}^2} + \sqrt{\pi_{n,1}^2 + \pi_{n,0}^2 + \pi_{n+1,0}^2} = 2\sqrt{x_n}.
\]

(70) \hspace{1cm}

Furthermore, for \( n > n^* \), we have:

\[
x_n \leq ||A_n||^2 x_{n-1} \\
\leq \lambda^2 x_{n-1} \\
\leq \lambda^{2(n-n^*)} x_{n^*}
\]

(71)
Therefore:

\[ \mathbb{E}[N_{H,\infty}] = \sum_{n_H=0}^{\infty} n_H (\pi_{n_H,0} + \pi_{n_H,1}) \]

\[ \leq \sum_{n_H=0}^{n^*} n_H (\pi_{n_H,0} + \pi_{n_H,1}) + \sum_{n_H=n^*}^{\infty} n_H (\pi_{n_H,0} + \pi_{n_H,1}) \]

\[ \leq n^* + \sum_{n=n^*}^{\infty} n 2 \sqrt{x_n} \]

\[ \leq n^* + 2 \sqrt{x_{n^*}} \sum_{n=n^*}^{\infty} n \lambda^{n-n^*} \]

\[ \leq n^* + 2 \frac{\lambda}{\lambda^{n^*}} \left( \frac{\lambda^{n^*-1}(n^*(1-\lambda) + \lambda)}{(1-\lambda)^2} \right) \]

\[ \leq n^* \left( 1 + \frac{2}{(1-\lambda)} \right) + o(n^*) \]

(72)

Where we successively used the splitting of an infinite sum, equation (70), equation (71), and the fact that \( x_n^* \leq 1 \). Note that \( \lambda \) only depends on \( \theta \) and not on \( n^* \) or \( p_H \). Using Little’s law to derive the bound for \( w_H \), this concludes our proof. Note that our \( n^* \) depends on the value of \( \theta \): for \( \theta > 0.5 \), we have \( n^* = \frac{k}{p_H} \) whereas for \( \theta < 0.5 \) we have \( n^* = \frac{k'}{p_H} \).

G Easy-to-match agents’ waiting times

We report here simulation results for the average waiting times of easy-to-match agents under \textit{BilateralMatch} and \textit{ChainMatch} for uncorrelated arrivals as modeled in Section 4.1. Observe that the average waiting time of \( E \) agents are orders of magnitude lower than that of \( H \) agents. Under \textit{ChainMatch} the average waiting of \( E \) agents increases with \( \theta_H \) but has almost not effect from increasing \( \theta_E \). For the \textit{BilateralMatch} however, while waiting times still seem to decrease approximately like \( 1/\theta_H \), we observe a sharp jump in waiting times when \( \theta_E > \theta_H \).
Figure 11: Simulations of a pool with independent arrivals of $H$ and $E$ agents, using ChainMatch and parameters $p_H = 0.02, p_E = 0.8$. Left panel shows the waiting times of $E$ agents as a function of $\theta_H$, for varying $\theta_E$. Right panel shows waiting times for $E$ agents as a function of $\theta_E$, for varying $\theta_H$.

Figure 12: Simulations of a pool with independent arrivals of $H$ and $E$ agents, using BilateralMatch and parameters $p_H = 0.02, p_E = 0.8$. Left panel shows the waiting times of $E$ agents as a function of $\theta_H$, for varying $\theta_E$. Right panel shows waiting times for $E$ agents as a function of $\theta_E$, for varying $\theta_H$. 

57