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Simultaneous Ad Auctions

Itai Ashlagi
Sloan School of Management, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139,
iashlagi@mit.edu

Dov Monderer
Faculty of Industrial Engineering and Management, Technion, 32000 Haifa, Israel,
dov@ie.technion.ac.il

Moshe Tennenholtz
Microsoft Israel R&D Center, Herzeliya and Faculty of Industrial Engineering and Management,
Technion, 32000 Haifa, Israel, moshet@ie.technion.ac.il

We consider a model with two simultaneous VCG ad auctions A and B where each advertiser chooses to participate in a single ad auction. We prove the existence and uniqueness of a symmetric equilibrium in that model. Moreover, when the click rates in A are pointwise higher than those in B, we prove that the expected revenue in A is greater than the expected revenue in B in this equilibrium. In contrast, we show that this revenue ranking does not hold when advertisers can participate in both auctions.

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1. Introduction. Search engines and publishers conduct ad auctions for potentially every keyword. In an ad auction, advertisers compete over positions in the web page associated with the results of searching for the corresponding keyword. The advertisers submit bids, and the position of displayed ads on the web page is determined based on these bids. Moreover, an advertiser pays the search engine each time a user clicks on her ad, where the charged price is also based on the submitted bids. Due to their enormous impact on search engines' and advertisers' revenues and because of the important challenges they provide for auction designers and participants, ad auctions have become a central topic of study in economics, electronic commerce, and marketing.[1] However, previous research has yet to account for the fact that similar ad auctions are held simultaneously by different search engines. That is, an advertiser has to choose not only how to bid, but where to bid. This paper initiates research on this question by examining two simultaneous ad auctions.

We study a model in which two VCG ad auctions are conducted simultaneously and advertisers choose a single ad auction to run their ad campaign. We adopt the standard symmetric independent private-value model. In particular, the game we analyze has two stages. In the first stage, each advertiser, after observing her type, chooses her probability for selecting each auction. We refer to this probability as the advertiser's participation strategy. After the realization of the participation strategy, the advertiser chooses her bid. No information is revealed after the first stage, and therefore, for technical convenience the game can be viewed as having only one stage in which each advertiser chooses a participation strategy and two bids, one for each auction. Indeed, advertisers often tend to concentrate on only one search engine in certain keyword markets. One reason for this behavior is the burden that advertisers have in running and managing their campaigns for each search engine. Another reason arises from the lack of flexibility in copying advertising campaign data from some search engines to others (see e.g., Edelman [5] and [6]).[2]

A main issue is how the revenue of a given auction is effected by the existence of another auction; of particular interest is the relationship between the click rate values (search engines' popularity) and the corresponding ad auctions' revenue. Specifically, do higher click rates result in higher revenue? To answer this question, we first analyze the equilibria in our model of simultaneous ad auctions. Because reporting truthfully is a dominant strategy in a VCG ad auction (for any distribution over valuations and any number of bidders), we consider an equilibrium within each auction, at which all participating bidders bid their true valuation. This reduces the

[1] See, e.g., Varian [20], Lahaie [10], Edelman et al. [7], Borgs et al. [3], Mehta et al. [12], and Athey and Ellison [2].

[2] Our model of simultaneous ad auctions is not the only reasonable one. An interesting alternative model would incorporate costs for running ad campaigns. Another interesting model is one in which each bidder has a limited budget, and her strategic decision is how to split her budget between the auctions.

strategy set of every bidder to the set of participation strategies. We prove that the auction selection game has an essentially unique symmetric equilibrium, whose structure is analyzed. Particular cases are presented and discussed.

Search engines use ad auctions as one of their main revenue sources. Intuitively, because an advertiser is charged each time a user clicks on her ad, one would expect that higher click rates will result in higher revenues. This implies that search firms should care about providing effective search engines, yielding high traffic to their sites. Indeed, we prove that when auction A is stronger than auction B, in the sense that the click rates in A are pointwise higher than those in B, the expected revenue of A is higher than the expected revenue of B in the essentially unique symmetric equilibrium. However, we show that this seemingly intuitive result is a consequence of advertisers selecting a single auction; in particular, we show that if all advertisers participate in both auctions, then a stronger auction need not yield a higher revenue than a weaker one.[3]

Our existence and uniqueness results in this setting generalize those by Burguet and Sakovics [4], who defined and analyzed a setting in which participants can choose among two simultaneous identical second-price single-item auctions with potentially distinct reserve prices. Gavious [8] extends Burguet and Sakovics [4] by allowing each item to have a different quality and shows that a symmetric equilibrium exists.[4] Hence, our existence result generalizes this symmetric equilibrium existence result for multiple different items. Both Burguet and Sakovics [4] and Gavious [8] study further properties under equilibrium. The former studies which reserve price an auctioneer should choose in order to maximize his own revenue given the reserve price of the other auctioneer. The latter analyzes the efficiency under the symmetric equilibrium. In this work we are interested in comparing auctioneers' revenues.

It is important to note that this paper does not discuss strategic organizers' competition in auction design, although it opens the road for such a model. In general, such a competition is modeled as a two-stage game, where in the first stage every auction organizer chooses the auction to conduct, and in the second stage each of the bidders decides which auction to attend and how to bid in this auction. Such an approach was taken in a restricted symmetric single-item auction setup, e.g., in Burguet and Sakovics [4] and in Monderer and Tennenholtz [13]. However, as was already shown in Burguet and Sakovics [4], the above two-stage game does not always possess a pure subgame-perfect equilibrium (they showed the existence of a mixed-strategy equilibrium). These works as well as ours assume a fixed allocation rule (up to a reserve price). Therefore, most of the literature (McAfee [11], Peters [16], Peters and Severinov [17]) on competition in auction design in the single-item setup has dealt with a model with many auctions and derived results about the limit (partially strategic) behavior of the market, when the number of auctions' organizers and buyers is approaching infinity. This approach does not seem the right one in the ad auction market, where only a few auctions' organizers (search engines) control the market.

Finally, both Burguet and Sakovics [4] and Peters and Severinov [17] assumed that the sellers use a second-price auction allowing the sellers to choose only a reserve price, i.e., they assume that the allocation rule is fixed. McAfee [11] allowed sellers to choose from a more general class of mechanisms but in a large market of sellers. Only recently, Pai [15] relaxed this assumption in a setting with a finite number of sellers, and showed that a seller might not always prefer to adopt the allocation rule of the second-price auction.

The paper is organized as follows. In §2 we define the basic model of VCG ad auctions. In §3 we define the simultaneous ad auction model with two VCG ad auctions, and present our main results. The existence and uniqueness theorem is proved in §4, and examples are provided in §4.4. The theorem about revenue inequality is proved in §5. Section 6 provides further discussion.

2. Preliminaries—VCG ad auctions. There are n advertisers that we call *bidders*, $n \geq 2$; a generic bidder is denoted by i, $1 \leq i \leq n$. In an ad auction there is a seller who offers for sale k positions, $k \geq 1$; a generic position is denoted by j, $1 \leq j \leq k$. Each bidder can receive at most one position. Because the seller cannot sell more positions than the number of bidders, it is assumed that $n \geq k$. The positions are sold for a fixed period of time. For each position j there is a commonly known *click rate* $\alpha_j > 0$, which is interpreted as the expected number of visitors at that position. It is assumed that positions have distinct click rates, and without loss of generality it is assumed that $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$. For convenience, we add a dummy position, position $k+1$ with $\alpha_{k+1} = 0$. Hereafter, unless otherwise specified, the term "position" includes the dummy position. Let $K = \{1, \ldots, k\}$ be the set of nondummy positions.

[3] A relevant phenomenon was demonstrated for VCG combinatorial auctions with complete information, where it was shown that higher valuations may reduce revenue. See, e.g., Rastegari et al. [18] and the references therein. We also discuss in the last section how this phenomenon also applies in the generalized second-price ad auction (the ad auction used in practice) under complete information.

[4] Gavious [8] refers to the equilibrium as a separating equilibrium.

If bidder i holds a position, every visitor to this position gives i a revenue of $v_i \in [0, 1]$, where v_i is called the *valuation* of i. We assume that for each i, v_i is private information and drawn independently from a commonly known distribution F defined on $[0, 1]$. Unless otherwise specified, we assume that F is *standard*, i.e., it is differentiable, it has a positive derivative on $[0, 1]$, and it has a density f.

It is assumed that bidders utility functions are quasi-linear. That is, if bidder i is assigned to position j and pays p per click, her utility is $\alpha_j(v_i - p)$.

Given the above, an ad auction mechanism is defined by its allocation rule and payment scheme. Each bidder i is required to submit a bid $b_i \in [0, 1]$. By $b_{(j)}$ we denote the jth-highest bid. Given the bids, the allocation rule determines the allocation of positions to the bidders, and the payment scheme determines how much each bidder will pay per click. In this paper we will consider only VCG ad auctions.

DEFINITION 2.1 (VCG AD AUCTION). An ad auction is called a *standard VCG ad auction* or, in short, *a VCG ad auction* if:

(i) The highest bidder receives the first position, the second-highest bidder receives the second position, and so on. Each bidder that does not receive a position $j \in K$ is assigned to the dummy position, $k+1$. Ties are resolved by the following simple priority rule over bidders: $i < t$ implies that i has priority over t whenever they submit the same bid.[5]

(ii) For any bid profile $\mathbf{b} = (b_1, \ldots, b_n)$ the bidder that is assigned to position $j \in K$ pays per each click

$$\frac{1}{\alpha_j} \sum_{l=\min(j+1,\, n)}^{\min(k+1,\, n)} b_{(j)}(\alpha_{j-1} - \alpha_j), \tag{1}$$

and all bidders that are assigned to the dummy position pay 0.[6]

A VCG ad auction with k nondummy positions and a click-rates vector $\alpha = (\alpha_1, \ldots, \alpha_k)$ is denoted by $G(k, \alpha)$. In some discussions, some of the parameters in $G(k, \alpha)$ whose values are obvious are omitted.

A VCG ad auction $G(k, \alpha)$, together with F, induces a Bayesian game G. A *strategy* for bidder i at this game is a function d_i: $V_i \to B_i$ that assigns a bid $d_i(v_i)$ to every possible value v_i of i. The expected utility for bidder i whose valuation is v_i at a profile of strategies $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is denoted by $U_i(v_i, \mathbf{d})$. For simplicity, we avoid here the explicit definition of U_i (which is the integral of the utility of i with respect to the joint distribution of all other bidders's valuations but i). Throughout the paper we use the standard notation regarding the subscript $-i$, e.g., $\mathbf{d}_{-i}$ denotes the bidding strategies of all bidders. A strategy profile $\mathbf{d}$ is a *Bayesian equilibrium* in G if for every bidder i and every v_i, and every strategy d'_i

$$U_i(v_i, \mathbf{d}) \geq U_i(v_i, d'_i, \mathbf{d}_{-i}).$$

It is well known that bidding truthfully is a weakly dominant strategy in a VCG ad auction (regardless of the value distributions of all players, even if the distributions are not identical). In particular, it is a best response to any bidding strategy profile, again regardless of other players' value distributions. Therefore, the truth-telling strategy profile is a Bayesian equilibrium in the Bayesian game induced by a VCG ad auction $G(k, \alpha)$ and any distribution of valuations of the players.

3. Simultaneous ad auctions—Main results.

Consider two VCG ad auctions, $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$, and recall the assumption that $n \geq k^A$ and $n \geq k^B$. In what follows, whenever necessary, it is assumed that $\alpha_j = 0$ for every $j > k^A$ and that $\beta_j = 0$ for every $j > k^B$. Auctions A and B form a Bayesian game for the bidders $1, 2, \ldots, n$, denoted by $H = H(A, B, F)$. In this game, each bidder simultaneously chooses one auction to participate in, and how to bid at each auction. Bidders can use mixed strategies to select an auction. We assume that a bidder cannot attend both auctions. This assumption captures single-campaign advertisers who run their campaign with only a single search engine. Because bidders can always guarantee a nonnegative utility by bidding zero, there is no harm in assuming that bidders always choose to participate in some auction. A strategy consists of a participation strategy and a bidding strategy. A participation strategy induces a distribution over valuations for the player. Hence, given any participation strategy profile, truth telling is an equilibrium in each auction. Given that, a *strategy* for a bidder i in H is now a tuple $\mathbf{q}_i = (q_i^A, q_i^B)$.[7] More precisely, q_i^L: $[0, 1] \to [0, 1]$, a Borel measurable function, is the probability that i will attend the auction L,

[5] Our results hold for every tie-breaking rule, including randomized ones.

[6] In a nonstandard VCG ad auction, every bidder may pay an additional amount depending only on other bidders' bids.

[7] In general, a strategy in H should also include the bidding strategy in each auction.

$L \in \{A, B\}$. In particular, $q_i^A(v_i) = 1 - q_i^B(v_i)$ for every valuation v_i. The expected utility for a bidder i whose value is v_i in H at a strategy profile $\mathbf{q} = (q_1, \ldots, q_n)$ is now naturally defined.

Because the distribution functions of all bidders are identical, it is natural to focus on symmetric strategies/equilibrium, and therefore we omit the bidder's index from a strategy. Hence, we refer to a strategy of a bidder as a vector $\mathbf{q} = (q^A, q^B)$. With some abuse of notation, we will also denote by $\mathbf{q}$ the strategy profile in which all bidders use the strategy (q^A, q^B).

3.1. Existence and uniqueness. Let $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$ be two VCG ad auctions, and let F be a distribution function. Let $\mathbf{q}$ be a symmetric equilibrium strategy. Note that changing $\mathbf{q}$ for a set of valuations v with $F-$probability 0 for which $0 < q^A(v) < 1$—does not change the utilities of any of the bidders. We therefore say that the game $H(A, B, F)$ possesses an *essentially unique* symmetric equilibrium if it possesses a symmetric equilibrium and for every two symmetric equilibria $\mathbf{q}, \hat{\mathbf{q}}$,

$$\mathbf{q}(v) = \hat{\mathbf{q}}(v) \quad F\text{-almost everywhere in } [0, 1].$$

We prove the existence and uniqueness of a symmetric equilibrium in $H(A, B, F)$:

THEOREM 3.1. *Let $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$ be a pair of VCG ad auctions.*
The game $H(A, B, F)$ possesses an essentially unique symmetric equilibrium. Moreover, if $\alpha_1 \geq \beta_1$ there exists a unique $0 \leq v^* \leq 1$ for which there exists a symmetric equilibrium $\mathbf{q} = (q^A, q^B)$ with the following properties:

$$\begin{cases} 0 < q^B(v) < 1 & \text{for every } 0 < v < v^*; \\ q^B(v) = 0 & \text{for every } v^* < v \leq 1. \end{cases} \tag{2}$$

Furthermore, $v^* = 0$ if and only if $\alpha_n \geq \beta_1$, and $v^* = 1$ if and only if $\alpha_1 = \beta_1$.

The condition $\alpha_1 \geq \beta_1$ in Theorem (3.1) is without loss of generality; otherwise, switch the names of the auctions.

The intuition for the existence result and (2) is the following. Similarly to Myerson [14], the expected utility for a bidder with valuation v in auction A (B) will be shown to be the integral of her expected click rate in auction A (B) over all values below her value v. Furthermore, because $\alpha_1 \geq \beta_1$ there exist a minimal value, namely v^*, at which a bidder with valuation $v \geq v^*$ who chooses auction A will obtain in expectation at least β_1 clicks regardless of the participation strategy of all other bidders. Therefore, if one can find a participation strategy, namely q, at which the expected click rates in both auctions are identical at each value $v \leq v^*$ if all bidders use q, then such a strategy will induce an equilibrium; indeed, the expected utility for a bidder at any $v \leq v^*$ will be identical in both auctions, and a bidder with valuation $v > v^*$ will strictly prefer auction A.

In §4 we prove Theorem 3.1, and we provide tools for computing the cutting point v^* and the values of the probabilities below v^*. The proof uses techniques from Burguet and Sakovics [4]. Unfortunately, their proof cannot be used directly to prove our result because they heavily use the fact that there is a single item both in the existence and uniqueness parts. Using a similar proof Theorem 3.1 can be extended to the setting in which each auction has a reserve price (smaller than 1). In §4.4 we apply the tools developed through the proof in order to explicitly find and discuss the equilibrium in special cases.

3.2. Revenues. Let $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$ be two ad auctions. In our main result we compare the revenues between auctions A and B in the game $H(A, B, F)$ under the symmetric equilibrium established in Theorem 3.1. We say that A is *stronger* than B if $k^A \geq k^B$ and $\alpha_j \geq \beta_j$ for every $1 \leq j \leq k^A$, and at least one inequality is strict.

THEOREM 3.2. *Let $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$ be VCG ad auctions. If A is stronger than B, the expected revenue in A is greater than the expected revenue in B in the essentially unique symmetric equilibrium of the game $H(A, B, F)$.*

Theorem 3.2 is proved in §5. A similar result holds for the case in which auction A has reserve price r^A, auction B has reserve price r^B, and $r^A \leq r^B$ (A is the stronger auction). Given a reserve price $0 < r^B < 1$ identifying the least reserve price $r^A \geq r^B$ that yields A a higher revenue than B's is left as an open question.

Theorem 3.2 seems an intuitive result. Consider, however, the following situation where a single seller conducts an auction with two bidders and two substitutable goods, where one good has lower quality than the other.

By raising the quality of the inferior good, the supply of quality "increases," which can yield lower revenue.[8] Thinking of the click rate as the quality of the position, a similar phenomenon can occur in ad auctions, i.e., *more clicks can yield lower revenue*. We formalize this.

Let $A = G(k, \alpha)$ be a VCG ad auction. Let $\tilde{v}_1, \ldots, \tilde{v}_n$ be independent random variables distributed F, and let $\tilde{v}_{(t)}$ be the tth reverse-order statistics generated by these random variables. In particular, $\tilde{v}_{(1)} \geq \tilde{v}_{(2)} \geq \cdots \geq \tilde{v}_{(n)}$. Let $\tilde{v}_{(t)}$ be identically zero for $t > n$. Assuming that bidders are truth telling, by summing the payments of all bidders the expected revenue in A is

$$\sum_{j=1}^{k}(\alpha_j - \alpha_{j+1}) jE[\tilde{v}_{(j+1)}] = \alpha_1 E[\tilde{v}_{(2)}] + \sum_{j=2}^{k} \alpha_j (jE[\tilde{v}_{(j+1)}] - (j-1)E[\tilde{v}_{(j)}]). \tag{3}$$

Observe from the right-hand side of (3) that an increase in the click rate of position $j \geq 2$ can result in a decrease in the revenue for A (this depends on the distribution of valuations F).

In particular, consider a setting in which all bidders participate in both auctions, A and B. Even if A is stronger than B, the revenue in A can be lower than the revenue in B. Theorem 3.2 asserts that when bidders choose a single auction to participate in, this phenomenon cannot occur.

4. Theorem 3.1: Proof and examples. Consider the game $H(A, B, F)$, where A and B are VCG auctions. Without loss of generality, assume that $\alpha_1 \geq \beta_1$.

If $\alpha_n \geq \beta_1$, then in particular $\alpha_n > 0$ and therefore $k^A \geq n$. In such a case, for an arbitrary bidder i, independently of all other bidders' strategies, the maximal utility in B, $\beta_1 v_i$ does not exceed her minimal utility in A, $\alpha_n v_i$. The proof for this case is immediate and left for the reader. Therefore, throughout the proof we assume that $\alpha_n < \beta_1$. We begin by developing some useful propositions in the following subsection.

4.1. Preparations. Let $\mathbf{q}$ be a symmetric strategy. For an arbitrary bidder i, let

$$\varphi(v, \mathbf{q}) = \text{Prob}_F(v_i \geq v, \ i \text{ chooses to participate in } B).$$

That is,

$$\varphi(v, \mathbf{q}) = \int_v^1 q^B(x)\, dF(x). \tag{4}$$

When all other bidders but t use the strategy $\mathbf{q}$, bidder t should compare her utilities in A and in B. When she computes her utility in A, she faces a random number of participants. Equivalently, bidder t can consider lack of participation in A as participation in A of a bidder with valuation 0. Hence, bidder t can assume that there exist exactly additional $n-1$ bidders in A such that the distribution function of each of them is F_q^A, where

$$F_q^A(v) = \varphi(v, \mathbf{q}) + F(v). \tag{5}$$

Similarly, for auction B, let $\psi(v, \mathbf{q}) = \int_v^1 q^A(x)\, dF(x)$, and let $F_q^B(v) = \psi(v, \mathbf{q}) + F(v)$. Because $q^A(v) = 1 - q^B(v)$ for all v, $\psi(v, \mathbf{q}) = 1 - F(v) - \varphi(v, \mathbf{q})$, and therefore

$$F_q^B(v) = 1 - \varphi(v, \mathbf{q}). \tag{6}$$

Denote by $P^A(v, \mathbf{q})$ ($P^B(v, \mathbf{q})$) the expected total payment in A (B) experienced by a bidder with valuation v given that each of the other bidders uses the strategy $\mathbf{q}$. Similarly, denote by $Q^A(v, \mathbf{q})$ ($Q^B(v, \mathbf{q})$) the expected click rate in A (B), and denote by $U^A(v, \mathbf{q})$ ($U^B(v, \mathbf{q})$) the expected utility in A (B). Obviously,

$$U^L(v, \mathbf{q}) = vQ^L(v, \mathbf{q}) - P^L(v, \mathbf{q}), \quad L \in \{A, B\}, \ v \in [0, 1]. \tag{7}$$

Note that a bidder with valuation v obtains position j in auction A if there are exactly $n-j$ other bidders, each of whom has a lower valuation than v in A and there are exactly $j-1$ bidders, each of whom has a higher valuation than v in A. Because ties have probability zero, the probability that the bidder obtains j and the above condition is not satisfied equals 0. Therefore,

$$Q^A(v, \mathbf{q}) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (F_q^A(v))^{n-j} (1 - F_q^A(v))^{j-1}. \tag{8}$$

[8] Independently, Gomes and Sweeney [9] discuss similar phenomena regarding sellers' incentives to "reduce" click rates.

Similarly,

$$Q^B(v, \mathbf{q}) = \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} (F_q^B(v))^{n-j} (1 - F_q^B(v))^{j-1}. \tag{9}$$

We need the following proposition whose proof is standard in mechanism design theory.

Proposition 4.1. *For every $L \in \{A, B\}$ and every symmetric strategy $\mathbf{q}$:*
1. *$Q^L(\cdot, \mathbf{q})$ is nondecreasing and continuous in* $[0, 1]$.
2. *For every symmetric strategy $\mathbf{q}$ and for every $v \in [0, 1]$,*

$$U^L(v, \mathbf{q}) = \int_0^v Q^L(x, \mathbf{q})\, dx. \tag{10}$$

Consequently, because Q^L is continuous, the derivative of $U^L(\cdot, \mathbf{q})$ equals $Q^L(\cdot, \mathbf{q})$ everywhere in $[0, 1]$.

Proof. We prove the proposition for $L = A$.

1. Let $v, w \in [0, 1]$. Because truth telling is a dominant strategy for a bidder, bidding v when her valuation equals v yields at least as bidding w, that is, $vQ^A(v, \mathbf{q}) - P^A(v, q) \ge vQ^A(w, \mathbf{q}) - P^A(w, \mathbf{q})$. Similarly, $wQ^A(w, \mathbf{q}) - P^A(w, \mathbf{q}) \ge wQ^A(v, \mathbf{q}) - P^A(v, \mathbf{q})$. Combining these inequalities yields

$$(v - w)(Q^A(v, \mathbf{q}) - Q^A(w, \mathbf{q})) \ge 0 \quad \text{for every } v, w \in [0, 1], \tag{11}$$

which implies that Q^A is nondecreasing. Because F_q^A is continuous in $[0, 1]$, Q^A is continuous as well by (8).

2. Let $v, w \in [0, 1]$. Recall that $U^A(v, \mathbf{q}) = vQ^A(v, \mathbf{q}) - P^A(v, \mathbf{q})$. Therefore, by the two inequalities we derived in part 1 of this proof, and by (7),

$$U^A(v, \mathbf{q}) - U^A(w, \mathbf{q}) \ge Q^A(w, \mathbf{q})(v - w).$$

By Rockafellar [19], this implies that $U^A(\cdot, \mathbf{q})$ is a convex function whose derivative equals Q^A almost everywhere in $[0, 1]$, and because $U^A(0, \mathbf{q}) = 0$, the required integral equality follows. □

The following functions are extensively used in our proofs. For every $0 \le x \le 1$ and every $0 \le y \le 1$, let

$$\tilde{Q}^A(x, y) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} (x + y)^{n-j} (1 - x - y)^{j-1}, \tag{12}$$

$$\tilde{Q}^B(x) = \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} (1 - x)^{n-j} x^{j-1}, \tag{13}$$

and let

$$Q(x, y) = \tilde{Q}^A(x, F(y)) - \tilde{Q}^B(x). \tag{14}$$

Note that by (8), (9), (5), and (6), for every $0 \le v \le 1$,

$$\text{(i)}\ \ Q^A(v, \mathbf{q}) = \tilde{Q}^A(\varphi(v, \mathbf{q}), F(v)); \qquad \text{(ii)}\ \ Q^B(v, q) = \tilde{Q}^B(\varphi(v, \mathbf{q})); \tag{15}$$

and therefore

$$Q(\varphi(v, \mathbf{q}), v) = Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}). \tag{16}$$

For a function $\phi(x, y)$ we denote by ϕ_x, ϕ_y the derivatives with respect to the first and second variable, respectively. Similarly, if $\phi(x)$ is a function of one variable, ϕ_x denotes the derivative of ϕ. The following technical lemma will be useful for us.

Lemma 4.1. (i) $\tilde{Q}^A_y(x, y) = \tilde{Q}^A_x(x, y)$ *for every* x, y.
(ii) $\tilde{Q}^A_x(x, y) > 0$ *for every* x, y *for which* $0 < x + y < 1$.
(iii) $\tilde{Q}^B_x(x) < 0$ *for every* $0 < x < 1$.
Consequently,
(iv) $Q_x(x, y) > 0$ *for every* x, y *for which* $0 < x + F(y) < 1$ *or* $[x + F(y) = 1$ *and* $0 < x, y < 1]$.
(v) $Q_y(x, y) > 0$ *for every* x, y *for which* $0 < x + F(y) < 1$.

Proof. The equality (i) is obvious. (ii) Recall the standard convention that for nonnegative integers $a < b$, $\binom{a}{b} = 0$. Note that

$$\begin{aligned}\tilde{Q}^A_x(x,y) &= \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1}(n-j)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1}(j-1)(x+y)^{n-j}(1-x-y)^{j-2}\\ &= \sum_{j=1}^{k^A} \alpha_j \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=2}^{k^A} \alpha_j \binom{n-2}{j-2}(n-1)(x+y)^{n-j}(1-x-y)^{j-2},\end{aligned}$$

where the last equality follows because $\binom{n-1}{j-1}(n-j) = \binom{n-2}{j-1}(n-1)$ and $\binom{n-1}{j-1}(j-1) = \binom{n-2}{j-2}(n-1)$. Therefore,

$$\begin{aligned}&\tilde{Q}^A_x(x,y)\\ &= \sum_{j=1}^{k^A} \alpha_j \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1} - \sum_{j=1}^{k^A-1} \alpha_{j+1} \binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1}\\ &= \alpha_{k^A}\binom{n-2}{k^A-1}(n-1)(x+y)^{n-k^A-1}(1-x-y)^{k^A-1} + \sum_{j=1}^{k^A-1}(\alpha_j - \alpha_{j+1})\binom{n-2}{j-1}(n-1)(x+y)^{n-j-1}(1-x-y)^{j-1}.\end{aligned} \tag{17}$$

Assume that $0 < x + y < 1$, which implies, in particular, that both summands in *RHS*(17) are nonnegative. If $n > k^A$, the first summand in *RHS*(17) is positive. If $n = k^A$, $k^A \geq 2$, and because $\alpha_j - \alpha_{j+1} > 0$ for every $1 \leq j \leq k^A - 1$, the second summand in *RHS*(17) is positive. Therefore, *RHS*(17) > 0, which completes the proof of (ii). Note that $\tilde{Q}^B(x) = \widehat{Q}(1-x, 0)$, where $\widehat{Q}$ is defined as $\tilde{Q}^A$, except that we replace α_j with β_j for every relevant j and k^A with k^B. Therefore, (iii) follows from (ii). □

Proposition 4.2. (i) *There exists a unique valuation in* $[0, 1]$ *denoted by* v^* *for which* $Q(0, v^*) = 0$. *Moreover,* $v^* > 0$, *and* $v^* = 1$ *if and only if* $\alpha_1 = \beta_1$.

(ii) *There exists a unique function* h: $(0, v^*) \to [0, 1]$ *such that the following two conditions hold for every* $0 < v < v^*$:

$$0 \leq h(v) \leq 1 - F(v).$$

$$Q(h(v), v) = 0.$$

Moreover, this unique function denoted by h *satisfies* $0 < h(v) < 1 - F(v)$ *for every* $0 < v < v^*$.

(iii) h *is continuously differentiable and* $0 < -h'(v)/f(v) < 1$ *for every* $0 < v < v^*$, *where* f *is the density function*; *that is,* $f = F'$.

(iv) *The function* h *can be continuously extended to the closed interval,* $[0, v^*]$. *Denote this extension also by* h; h *satisfies* $h(v^*) = 0$.

Proof. (i) By part (v) of Lemma 4.1, $Q(0, v)$ is increasing in $v \in [0, 1]$. Because $Q(0, 0) = \alpha_n - \beta_1 < 0$ and $Q(0, 1) = \alpha_1 - \beta_1 \geq 0$, the requested results follow.

(ii) Let $v \in (0, v^*)$. Because $Q(0, y)$ is increasing in $y \in [0, 1]$ and $Q(0, v^*) = 0$, $Q(0, v) < 0$. By part (iv) in Lemma 4.1, $Q(x, v)$ is increasing in $x \in [0, 1 - F(v)]$, and therefore the proof is completed if we show that $Q(1 - F(v), v) > 0$. Indeed, $Q(1 - F(v), v) = \tilde{Q}^A(1 - F(v), F(v)) - \tilde{Q}^B(1 - F(v))$. However, $\tilde{Q}^A(1 - F(v), F(v)) = \alpha_1$ and by part (iii) of Lemma 4.1, $\tilde{Q}^B(1 - F(v)) < \tilde{Q}^B(0) = \beta_1$. Therefore, $Q(1 - F(v), v) > \alpha_1 - \beta_1 \geq 0$, which completes the proof of this part.

(iii) Let $0 < v < v^*$. Because $Q(h(v), v) = 0$ and by part (iv) in Lemma 4.1, $Q_x(h(v), v) > 0$, the implicit function theorem implies that there exists an interval $(v - \delta, v + \delta)$ around v and a unique real-valued function, g defined on this interval such that $g(v) = h(v)$ and $Q(g(y), y) = 0$ for every y in this interval. Moreover, g is continuously differentiable in this interval. For sufficiently small $\delta > 0$, $g(y)$ is sufficiently close to $g(v) = h(v)$, and because $0 < h(v) < 1 - F(v)$, $g(y) \in [0, 1 - F(v)]$. Therefore, by what we proved in the previous part of this theorem, $g(y) = h(y)$ for every y in this smaller neighborhood of v. This implies that h is continuously differentiable in this smaller neighborhood of v, and in particular it is differentiable in v. By the implicit function theorem,

$$h'(v) = -\frac{Q_y(h(v), v)}{Q_x(h(v), v)}.$$

By parts (iv) and (v) of Lemma 4.1, $-h'(v) > 0$. It remains to prove that $-h'(v)/f(v) < 1$. That is, we have to prove that

$$\frac{Q_y(h(v), v)}{Q_x(h(v), v) f(v)} < 1. \tag{18}$$

Note that $Q_y(h(v), v) = \tilde{Q}^A_y(h(v), F(v)) f(v)$, and by part (i) of Lemma 4.1, $\tilde{Q}^A_y(h(v), F(v)) f(v) = \tilde{Q}^A_x(h(v), F(v)) f(v)$. Also, $Q_x(h(v), v) = \tilde{Q}^A_x(h(v), F(v)) - \tilde{Q}^B_x(h(v)) > \tilde{Q}^A_x(h(v), F(v))$ by part (iii) of Lemma 4.1. Hence, (18) holds.

(iv) We first prove that $\lim_{v\to v^*} h(v) = 0$. Indeed, because $h'(v) < 0$ for $0 < v < v^*$, h is decreasing in $(0, v^*)$, and because in addition $h(v)$ is bounded below by 0, there exists $c \geq 0$ such that $\lim_{v\to v^*} h(v) = c$. We proceed to prove that $c = 0$. Because $Q(h(v), v) = 0$ for every $0 < v < v^*$, $Q(c, v^*) = 0$. Moreover, $0 \leq c \leq 1 - F(v^*)$. Hence, $c = 0$ if $v^* = 1$. Consider the case $v^* < 1$: because by part (iv) of Lemma 4.1, $Q(x, v^*)$ is increasing in $x \in [0, c]$, and $Q(0, v^*) = 0$, it must be that $c = 0$.

Similarly, because h is decreasing and bounded from above in $(0, v^*)$, the limit, $\lim_{v\to 0} h(v)$ exists and is denoted by $h(0)$. □

We end this subsection with the following useful lemma:

Lemma 4.2. (i) *For every* $v \in [0, v^*)$, $Q(0, v) < 0$; *for every* $v \in (v^*, 1]$, $Q(0, v) > 0$.

(ii) *For every* $v \in (v^*, 1]$ *and for every* $x \in [0, 1 - F(v)]$, $Q(x, v) > 0$.

(iii) *For every* $v > v^*$ *and for every strategy* $\mathbf{q}$, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) > 0$.

Proof. (i) By part (v) in Lemma 4.1, $Q(0, y)$ is increasing in $y \in [0, 1]$. Because $Q(0, v^*) = 0$, the result follows.

(ii) By part (ii) of Lemma 4.1, $Q(x, v)$ is increasing in $x \in [0, 1 - F(v)]$, and therefore $Q(x, v) \geq Q(0, v) > 0$ for every $x \in [0, 1 - F(v)]$.

(iii) By (16), $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = Q(\varphi(v, \mathbf{q}), v) > 0$ by the previous part because $\varphi(v, \mathbf{q}) \leq 1 - F(v)$. □

We are now ready to prove the existence and uniqueness of symmetric equilibrium.

4.2. Existence. In this section we prove by construction that a symmetric equilibrium exists. We define $\tilde{\mathbf{q}} = (\tilde{q}^A, \tilde{q}^B)$ as follows (recall that $\tilde{q}^B = 1 - \tilde{q}^A$): the values, $\tilde{q}^B(0)$, $\tilde{q}^B(v^*)$ are left unspecified, and for other $0 < v \leq 1$,

$$\tilde{q}^B(v) = \begin{cases} -\dfrac{h'(v)}{f(v)} & 0 < v < v^*; \\ 0 & v^* < v \leq 1. \end{cases} \tag{19}$$

In order to prove that $\tilde{\mathbf{q}}$ is a symmetric equilibrium, it suffices to prove the following two claims:

(a) $U^A(v, \tilde{\mathbf{q}}) - U^B(v, \tilde{\mathbf{q}}) = 0$ for every $0 \leq v \leq v^*$;

(b) $U^A(v, \tilde{\mathbf{q}}) - U^B(v, \tilde{\mathbf{q}}) > 0$ for every $v > v^*$.

We first compute the function $\varphi(v, \tilde{\mathbf{q}})$. Obviously, for every $v > v^*$, $\varphi(v, \tilde{\mathbf{q}}) = \int_v^1 \tilde{q}^B(x) f(x)\, dx = 0$. We will show that:

$$\varphi(v, \tilde{\mathbf{q}}) = h(v) \quad \text{for every } 0 \leq v \leq v^*. \tag{20}$$

Indeed, if $v = v^*$ the proof is obvious because $\varphi(v^*, \tilde{\mathbf{q}}) = 0 = h(v^*)$. Then let $0 \leq v < v^*$. For sufficiently small $\epsilon > 0$, $\varphi(v, \tilde{\mathbf{q}}) = I(\epsilon) + \int_{v+\epsilon}^{v^*-\epsilon} \tilde{q}^B(x) f(x)\, dx + J(\epsilon)$, where $I(\epsilon)$ equals the integral over the interval $[v, v+\epsilon]$ and $J(\epsilon)$ equals the integral over the interval $[v^* - \epsilon, v^*]$. Therefore, $\varphi(v, \tilde{\mathbf{q}}) = I(\epsilon) + h(v+\epsilon) - h(v^* - \epsilon) + J(\epsilon)$. Because h is continuous in $[0, v^*]$ and $I(\epsilon), J(\epsilon)$ converges to zero when $\epsilon \to 0$, $\varphi(v, \tilde{\mathbf{q}}) = h(v) - h(v^*)$, and because by part 4 in Proposition 4.2 $h(v^*) = 0$, (20) holds. Because $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = Q(\varphi(v, \tilde{\mathbf{q}}), v)$, by (20), $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = Q(h(v), v)$ for every $0 \leq v \leq v^*$. Therefore, because by Proposition 4.2 $Q(h(v), v) = 0$ for every $0 \leq v \leq v^*$, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = 0$ for every $0 \leq v \leq v^*$. Therefore, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = \int_0^v (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))\, dx = 0$ for every $0 \leq v \leq v^*$, which proves (a). By Lemma 4.2, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) > 0$ for every $v > v^*$. Hence, for every $v > v^*$, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = \int_{v^*}^v (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))\, dx > 0$ by Lemma 4.1, which proves (b).

4.3. Uniqueness. Let $\tilde{\mathbf{q}}$ be the symmetric equilibrium defined (19). In this section we will prove that $\tilde{\mathbf{q}}$ is an essentially unique symmetric equilibrium. Let $\mathbf{q}$ be any other symmetric equilibrium. We first prove that:

$$\begin{cases} U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0 & \text{for every } v \in [0, v^*]; \\ U^A(v, \tilde{\mathbf{q}}) - U^B(v, \mathbf{q}) > 0 & \text{for every } v > v^*. \end{cases} \tag{21}$$

Establishing (21) proves the essential uniqueness as follows: In the interval $(v^*, 1)$ a bidder in equilibrium must choose A, and therefore $q^B(x) = 0 = \tilde{q}^B(x)$ for every $x > v^*$ (recall, considering the other interval, the derivative of $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q})$ equals zero almost everywhere in $(0, v^*)$. Therefore, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = 0$ almost everywhere in this interval. Because $Q^A(v, \mathbf{q})$ and $Q^B(v, \mathbf{q})$ are continuous in v, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = 0$ for every $v \in [0, v^*]$. Therefore, $Q(\varphi(v, \mathbf{q}), v) = 0$ for every $v \in [0, v^*]$, and because $\varphi(v, \mathbf{q}) = h(v)$ for every $v \in [0, v^*]$, it must be that $\varphi(v, \mathbf{q}) = \varphi(v, \tilde{\mathbf{q}})$ in this interval. This finally implies that $q^B(x) = \tilde{q}^B(x)$ almost everywhere in $[0, v^*]$.

In order to prove (21), we need the following technical lemma, the proof of which is standard and hence omitted.

Lemma 4.3. *Let ϕ: $[a, b] \to R$ be a continuous function with $\phi(a) = \phi(b) = 0$ that satisfies the following property: for every $a \le c < d \le b$ for which $\phi(c) = \phi(d) = 0$ there exists $c < z < d$ such that $\phi(z) = 0$. Then, $\phi(x) = 0$ for every $a \le x \le b$.*

Lemma 4.4. *Let $0 \le c < d \le 1$ be two valuations for which $U^A(c, \mathbf{q}) - U^B(c, \mathbf{q}) = 0$ and $U^A(d, \mathbf{q}) - U^B(d, \mathbf{q}) = 0$. Then, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0$ for every $c \le v \le d$.*

Proof. Let $\phi(v) = U^A(v, \mathbf{q}) - U^B(v, \mathbf{q})$. By Lemma 4.3 it suffices to prove that there exists $c < z < d$ for which $\phi(z) = 0$. Assume in negation that such z does not exist. Therefore, either $\phi(v) > 0$ for every $c < v < d$ or $\phi(v) < 0$ for every $c < v < d$. Without loss of generality, $\phi(v) < 0$ for every $c < v < d$. Note that because $\phi(v) < 0$ for every $v \in (c, d)$, $q^B(v) = 1$ for every such v. Therefore, $Q^B(d, \mathbf{q}) - Q^B(c, \mathbf{q}) = \tilde{Q}^B(\varphi(d, \mathbf{q})) - \tilde{Q}^B(\varphi(c, \mathbf{q})) = \tilde{Q}^B(\varphi(d, \mathbf{q})) - \tilde{Q}^B(\varphi(d, \mathbf{q}) + F(d) - F(c))$. Because $\tilde{Q}^B$ is decreasing in $[0, 1]$,

$$Q^B(d, \mathbf{q}) > Q^B(c, \mathbf{q}). \tag{22}$$

Recall that by Lemma 4.1, $U^B(v, \mathbf{q}) = U^A(c, \mathbf{q}) + \int_c^v Q^A(x, \mathbf{q})\, dx$, and similarly, $U^B(v, \mathbf{q}) = U^B(c, \mathbf{q}) + \int_c^v Q^B(x, \mathbf{q})\, dx$. Therefore, $\phi(v) = \int_c^v (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))\, dx$. We claim that $Q^A(c, \mathbf{q}) \le Q^B(c, \mathbf{q})$. Indeed, if $Q^A(c, \mathbf{q}) - Q^B(c, \mathbf{q}) > 0$, for sufficiently small $\epsilon > 0$, $Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}) > 0$ for every $x \in [c, c + \epsilon]$. Therefore, for every $v \in (c, c + \epsilon]$, $\phi(v) > 0$, contradicting our negation assumption. Similarly, because $\phi(v) = -\int_v^d (Q^A(x, \mathbf{q}) - Q^B(x, \mathbf{q}))\, dx$, $Q^A(d, \mathbf{q}) \ge Q^B(d, \mathbf{q})$. Hence, we have:

$$Q^A(c, \mathbf{q}) \le Q^B(c, \mathbf{q}) < Q^B(d, \mathbf{q}) \le Q^A(d, \mathbf{q}). \tag{23}$$

However, because $q^A(v) = 0$ for every $c < v < d$, $Q^A(c, \mathbf{q}) = Q^A(d, \mathbf{q})$, contradicting (23). Therefore, there exists $c < z < d$ for which $\phi(z) = 0$. This completes the proof. □

Recall that $U^A(0, \mathbf{q}) - U^B(0, \mathbf{q}) = 0$. Define

$$d = \sup\{v \in [0, 1] \mid U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0\}. \tag{24}$$

By continuity, $U^A(d, \mathbf{q}) - U^B(d, \mathbf{q}) = 0$, and by Lemma 4.4, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0$ for every $0 \le v \le d$. We claim that if $d < 1$, then $U^A(v, \mathbf{q}) > U^B(v, \mathbf{q})$ for every $v > d$. By Lemma 4.4, either $U^A(v, \mathbf{q}) > U^B(v, \mathbf{q})$ or $U^A(v, \mathbf{q}) < U^B(v, \mathbf{q})$ for every $v > d$. Suppose in negation that $U^A(v, \mathbf{q}) < U^B(v, \mathbf{q})$ for all $v > d$. Hence, $q^B(v) = 1$ for every $v > d$. Therefore, by Lemma 4.1 $Q^B(v, \mathbf{q}) = \tilde{Q}^B(\varphi(v, \mathbf{q})) < \tilde{Q}^B(0) = \beta_1$. Therefore, because $Q^A(v, \mathbf{q}) = \alpha_1$ for every $v > d$, by Lemma 4.1 $U^A(v, \mathbf{q}) = U^A(d, \mathbf{q}) + \int_d^v Q^A(x, \mathbf{q}) = U^A(d, \mathbf{q}) + \alpha_1(v - d) \ge U^B(d, \mathbf{q}) + \beta_1(v - d) > U^B(d, \mathbf{q}) + \int_d^v Q^B(x, \mathbf{q}) = U^B(v, \mathbf{q})$, contradicting our negative assumption. Hence, in order to establish (21), it suffices to prove that $d = v^*$. Before we do it, note that because $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = 0$ for every $0 \le v \le d$, the derivative of $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q})$ equals 0 almost everywhere in this interval. Hence, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = 0$ almost everywhere. However, because Q^A, Q^B are continuous, the equality holds everywhere; that is,

$$Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) = 0 \quad v \in [0, d]. \tag{25}$$

Assume in negation that $v^* < d$. By Lemma 4.2, $Q^A(v, \mathbf{q}) - Q^B(v, \mathbf{q}) > 0$ for every $v > v^*$ contradicting (25).

Assume in negation that $d < v^*$. Let $d < z < v^*$. By Lemma 4.2, $Q(0, z) < 0$. On the other hand, $\varphi(z, \mathbf{q}) = \int_z^1 q^B(x)\, dF(x) = 0$ because for $x > d$, $q^B(x) = 0$. Therefore, $Q(\varphi(z, \mathbf{q}), v) = Q^A(z, \mathbf{q}) - Q^B(v, \mathbf{q}) = Q(0, z) < 0$. Let $d < v < v^*$, $U^A(v, \mathbf{q}) - U^B(v, \mathbf{q}) = \int_d^v (Q^A(z, \mathbf{q}) - Q^B(v, \mathbf{q}))\, dz < 0$, contradicting (25). This completes the proof. □

4.4. Examples for special cases. In this section we give one example where the symmetric equilibrium can be determined in closed form and another that shows that equilibrium participation strategies need not be monotone.

Example 4.1 ($k^A = k^B = 2$). First, we consider two ad auctions, each with two positions; that is, $k^A = k^B = 2$. In addition, it is assumed that $\alpha_1 > \beta_1 > \alpha_2$. When $n = 2$ the structure of equilibrium is revealed analytically.

By (12) and (13), $\tilde{Q}^A(x, y) = \alpha_1(x+y) + \alpha_2(1-x-y)$, and $\tilde{Q}^B(x) = \beta_1(1-x) + \beta_2(x)$. Recall that $Q(x, y) = \tilde{Q}^A(x, F(y)) - \tilde{Q}^B(x)$, and that v^* is the unique solution of $Q(0, v^*) = 1$. Hence, $v^* = F^{-1}((\beta_1 - \alpha_2)/(\alpha_1 - \alpha_2))$. The function $h(v)$ is determined by $Q(h(v), v) = 0$. Hence, $h(v) = [\beta_1 - \alpha_2 - (\alpha_1 - \alpha_2)F(v)]/(\alpha_1 - \alpha_2 + \beta_1 - \beta_2)$. Because $\tilde{q}^B(v) = -h'(v)/f(v)$, the essentially unique equilibrium $\tilde{\mathbf{q}}$ satisfies:

$$\tilde{q}^B(v) = \begin{cases} \dfrac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_2 + \beta_1 - \beta_2} & 0 < v < v^*; \\ 0 & v^* < v \leq 1. \end{cases} \tag{26}$$

Hence, bidders with high valuations participate with probability 1 in auction A, whereas a bidder with a low valuation randomizes and assigns a constant probability to each of the auctions at the interval $(0, v^*)$. Note that if $\alpha_1 - \alpha_2 > \beta_1 - \beta_2$, $(\alpha_1 - \alpha_2)/(\alpha_1 - \alpha_2 + \beta_1 - \beta_2) > 1/2$; that is, a bidder with a low valuation assigns a higher probability to the weaker auction.

When $n = 2$, the function $h(v)$ is determined by a polynomial equation in $h(v)$ of degree 1, and therefore $\tilde{q}^B(v)$ is constant at the first interval. However, for $n > 2$, $h(v)$ is determined by a polynomial equation of degree greater than 1, and $\tilde{q}^B(v)$ is not a constant function.

Example 4.2 (Strategies Are Not Monotone). It is interesting to note that $\tilde{q}^B(v)$ may be increasing or decreasing in $(0, v^*)$. This is shown in the examples illustrated in Figures 1 and 2, in which the equilibrium is computed by running a computerized method. In both examples $n = 4$, $k^A = k^B = 4$, and F is the uniform distribution. In Figure 1, the click-rate vectors of ad auctions A and B are $\alpha = (100, 70, 50, 20)$ and $\beta = (80, 30, 10, 5)$, respectively, and the cutting point is $v^* = 0.76$. In Figure 2 the click-rate vectors in ad auctions A and B are $\alpha = (90, 80, 60, 30)$ and $\beta = (85, 70, 40, 10)$, respectively, and the cutting point is $v^* = 0.85$.

5. Proof of Theorem 3.2.

Let $A = G(k^A, \alpha)$ and $B = G(k^B, \beta)$ be VCG ad auctions such that A is stronger than B. Let R^A (R^B) be the expected revenue in auction A (B) at the essentially unique symmetric equilibrium, $\tilde{\mathbf{q}}$ in $H(A, B, F)$. We have to prove that

$$R^A - R^B > 0.$$

Obviously, for $L \in \{A, B\}$, $R^L = n\int_0^1 P^L(v, \tilde{\mathbf{q}})\tilde{q}^L(v)f(v)\,dv$. In the rest of the proof, $\tilde{\mathbf{q}}$ is fixed and therefore omitted from the description of the functions. By the proof of Theorem 3.1 and by (7), $P^A(v) = P^B(v)$ for every $v \in [0, v^*]$, and because, in addition, $\tilde{q}^B(v) = 0$, $\tilde{q}^A(v) = 1$ for every $v^* \leq v \leq 1$.

$$\frac{R^A - R^B}{n} = \int_0^{v^*} P^A(v)(1 - 2\tilde{q}^B(v))f(v)\,dv + \int_{v^*}^1 P^A(v)f(v)\,dv.$$

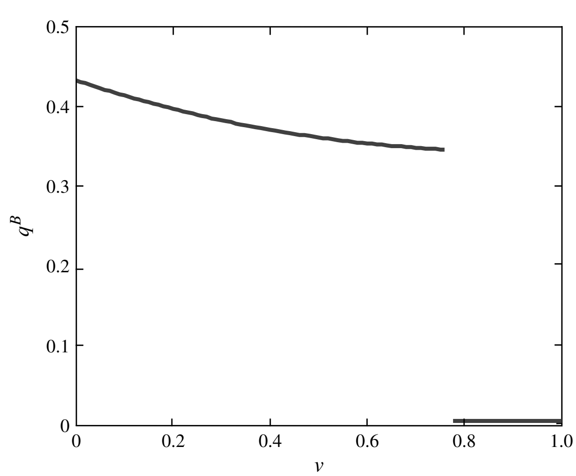

Figure 1. Probability for choosing auction B in the symmetric equilibrium when $n = 4$, $\alpha = (100, 70, 50, 20)$, and $\beta = (80, 30, 10, 5)$

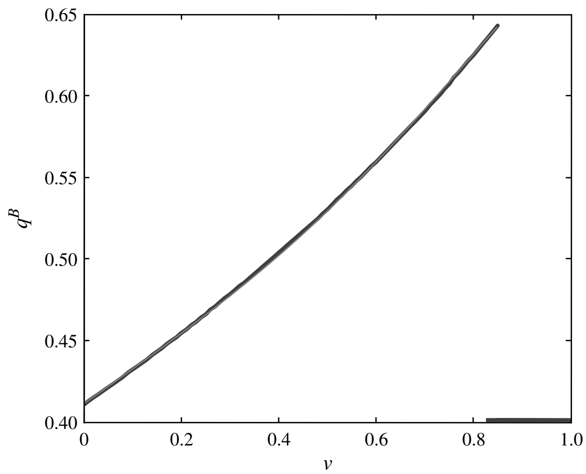

FIGURE 2. Probability for choosing auction B in the symmetric equilibrium when $n = 4$, $\alpha = (90, 80, 60, 30)$, and $\beta = (85, 70, 40, 10)$

Let $\Delta = (R^A - R^B)/n$. Because for $v > v^*$, $\tilde{q}^B(v) = 0$, $1 = 1 - 2\tilde{q}^B(v)$ for such v, and therefore

$$\Delta = \int_0^1 P^A(v)(1 - 2\tilde{q}^B(v)) f(v)\, dv.$$

Plug in $P^A(v) = vQ^A(v) - U^A(v)$ in the last equality to get

$$\Delta = \int_0^1 vQ^A(v)(1 - 2\tilde{q}^B(v)) f(v)\, dv - \int_0^1 U^A(v)(1 - 2\tilde{q}^B(v)) f(v)\, dv.$$

Because $U^A(v) = \int_0^v Q^A(x)\, dx$, the second term in the right-hand side of the last equality is a double integral. By changing the order of the integrals in this second term, and moving to the parameter x, we get

$$\Delta = \int_0^1 Q^A(x)[xf(x) - 2x\tilde{q}^B(x) f(x) - 1 + F(x) + 2\varphi(x)]\, dx. \tag{27}$$

We are about to apply the method of integration by parts to the right-hand side of (27). For that matter let $g(x) = xf(x) - 2x\tilde{q}^B(x) f(x) - 1 + F(x) + 2\varphi(x)$. Therefore,

$$\Delta = \int_0^1 Q^A(x) g(x)\, dx. \tag{28}$$

Let $G(x) = \int_0^x g(t)\, dt$. We claim that the derivative $Q^A_x(x)$ exists at each of the intervals $(0, v^*)$ and $(v^*, 1)$ and it is bounded at each of these intervals, and that $G(1) = 0$. Therefore, by integration by parts,

$$\Delta = -\int_0^1 Q^A_x(x) G(x)\, dx. \tag{29}$$

Indeed, for $0 < x < v^*$, $Q^A(x) = Q^B(x) = \tilde{Q}^B(h(x))$. Therefore, because h is continuously differentiable at that interval, $Q^A_x(x) = \tilde{Q}^B_x(h(x))h'(x)$. For $v^* < x < 1$, $Q^A(x) = \tilde{Q}^A(0, F(x))$, and therefore $Q^A_x(x) = \tilde{Q}^A_x(0, F(x)) f(x)$. It remains to show that $G(1) = 0$. Indeed, it is easily verified that $G(x) = xF(x) + 2x\varphi(x) - x$ for every $0 \le x \le 1$. Hence, $G(1) = 0$. Because by what we showed above $Q^A_x(x) > 0$ except for at most three values of x, in order to prove that $\Delta < 0$, it suffices to prove that

$$G(x) = xF(x) + 2x\varphi(x) - x < 0 \quad \text{for every } 0 < x < 1. \tag{30}$$

For every $v^* \le x < 1$, $G(x) = xF(x) - x < 0$. Therefore, it remains to prove (30) for $0 < x < v^*$. In order to prove it, we use the fact that A is stronger than B. By rearranging the terms in (30), we have to prove that

$$\varphi(x) < \frac{1 - F(x)}{2} \quad \text{for every } 0 < x < v^*. \tag{31}$$

Let $x \in (0, v^*)$ and let $\tilde{h}(x) = (1 - F(x))/2$. Note that $\tilde{h}(x) + F(x) = (1 + F(x))/2$ and $1 - \tilde{h}(x) - F(x) = (1 - F(x))/2$. Therefore, by (12) and (13)

$$\tilde{Q}^A(\tilde{h}(x), F(x)) = \sum_{j=1}^{k^A} \alpha_j \binom{n-1}{j-1} \left(\frac{1+F(x)}{2}\right)^{n-j} \left(\frac{1-F(x)}{2}\right)^{j-1} \tag{32}$$

and

$$\tilde{Q}^B(\tilde{h}(x)) = \sum_{j=1}^{k^B} \beta_j \binom{n-1}{j-1} \left(\frac{1+F(x)}{2}\right)^{n-j} \left(\frac{1-F(x)}{2}\right)^{j-1}. \tag{33}$$

Therefore, $Q(\tilde{h}(x), x) = \tilde{Q}^A(\tilde{h}(x), F(x)) - \tilde{Q}^B(\tilde{h}(x)) > 0$ because A is stronger than B. Let $0 < x < v^*$. Recall that the function $Q(v, x)$ is increasing in $0 \le v \le 1 - F(x)$. Because $h(x), \tilde{h}(x)$ are in this interval, and $Q(h(x), x) = 0 < Q(\tilde{h}(x), x)$, $h(x) < \tilde{h}(x)$. Because in $(0, v^*)$, $h(x) = \varphi(x)$, (31) follows. □

6. Discussion. In this paper we analyzed a model in which two VCG ad auctions are conducted simultaneously and advertisers need to choose a single auction in which to participate. Finding weak conditions under which our results hold for other ad auctions is an interesting task. A natural way to begin tackling this problem is by using revenue equivalence theorems. In existing systems, auctions' organizers run variants of the GSP ad auction. This auction has the same allocation rule as the VCG ad auction and its payment scheme is as follows: the bidder that is assigned to position j pays per click the bid of the bidder that is in position $j+1$. Recently, it has been shown by Gomes and Sweeney [9] that in the GSP ad auction under incomplete information the existence of an efficient equilibrium depends on the click rates. Thus, extending Theorems 3.1 and 3.2 to GSP ad auctions is already an intriguing task.

We have shown that higher click rates do not necessarily imply higher revenue in a VCG ad auction. As it turns out, Varian [20] and Edelman et al. [7] showed that with complete information, the VCG outcome is obtained in an equilibrium of the generalized second-price (GSP) ad auction. In addition, it was empirically claimed by Varian [20] that in practice the equilibrium that generates the VCG outcome is likely to be played.[9] Therefore, under complete information higher click rates can yield lower revenue in the GSP ad auction.

The implications of these findings for a search engine are as follows: a search engine should take into careful consideration the characteristics of the positions in the search results because modifying click rates (for example by making some positions less attractive) can improve revenue. Moreover, by attracting advertisers and optimizing the mechanism it may be possible for an ad auction to outperform the revenue of a stronger ad auction with higher click rates.

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