Instructions: Before starting this assignment, please read the section on “Homework” at http://web.stanford.edu/class/math61cm/61cm-info.pdf

1. Use mathematical induction to check that \( \sum_{j=1}^{n} j2^j = (n-1)2^{n+1} + 2 \) for \( n = 1, 2, \ldots \).

Note: The principle of mathematical induction is one of the basic properties of the natural numbers \( \mathbb{N} = \{1, 2, \ldots \} \), and we shall use it frequently; it says that if a subset \( P \subset \mathbb{N} \) has the properties (a) \( 1 \in P \), and (b) \( n \in P \Rightarrow n+1 \in P \) for each \( n = 1, 2, \ldots \), then \( P = \mathbb{N} \). Slightly more concretely, that is the same as saying that if \( P_n \) is a (true or false) proposition for each \( n = 1, 2, \ldots \) and if (a) \( P_1 \) is true, and (b) for each \( n \) we are able to check that \( P_{n+1} \) is true whenever \( P_n \) is true, then \( P_n \) is true for all \( n = 1, 2, \ldots \).

2. Prove that \( 1 + \sqrt{2} \) is irrational (i.e. not rational).

(Recall that \( x \) is rational means that \( \exists \) integers \( p, q \) with \( q \neq 0 \) such that \( x = p/q \).)

Hint: First show that \( 1 + \sqrt{2} \) is rational if and only if \( \sqrt{2} \) is rational. If \( \sqrt{2} \) is rational, then we can write \( \sqrt{2} = p/q \), where \( p, q \) are positive integers without common factors.

3. Prove or give a counterexample to the following:

Claim: Let \( V \) be a vector space over a field \( F \), and suppose that \( W_1, W_2 \) and \( W_3 \) are subspaces of \( V \). Suppose also that \( W_1 + W_3 = W_2 + W_3 \). Then \( W_1 = W_2 \).

4. Give an example of a subset \( U \subset \mathbb{R}^2 \) such that \( U \) is closed under scalar multiplication but is not a subspace.

5. Let \( V \) be a vector space over a field \( F \) and \( U, W \) be subspaces of \( V \). Suppose \( U \cup W \) is also a subspace. Show that either \( U \subseteq W \) or \( W \subseteq U \).

6. Let \( a, b, c, d \) be elements of a field \( F \), and consider the system of equations

   \[
   ax + by = 0, \quad cx + dy = 0
   \]

Show that \( x = y = 0 \) is the only solution if and only if \( ad - bc \neq 0 \).

7. Using the dot product, prove, for any vectors \( \underline{x}, \underline{y} \in \mathbb{R}^n \):

   (a) The parallelogram law: \( \|\underline{x} - \underline{y}\|^2 + \|\underline{x} + \underline{y}\|^2 = 2(\|\underline{x}\|^2 + \|\underline{y}\|^2) \)

   (b) The law of cosines: \( \|\underline{x} - \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 - 2\|\underline{x}\|\|\underline{y}\|\cos \theta \), assuming \( \underline{x}, \underline{y} \) are non-zero and \( \theta \) is the angle between \( \underline{x} \) and \( \underline{y} \) as discussed in lecture.

   (c) Give a geometric interpretation of identities (a) and (b).

   That is, describe what (a) is saying about the parallelogram determined by \( \underline{x}, \underline{y} \)—i.e. \( OACB \) where \( \overrightarrow{OA} = \underline{x}, \overrightarrow{OB} = \underline{y}, \overrightarrow{OC} = \underline{x} + \underline{y}, \) and what (b) is saying about the triangle determined by \( \underline{x} \) and \( \underline{y} \)—i.e. \( OAB \), where \( \overrightarrow{OA} = \underline{x}, \overrightarrow{OB} = \underline{y}. \)
8. Give a bijective proof that \( \binom{k}{r} \binom{n}{k} = \binom{n}{r} \binom{n-r}{k-r} \).

9. Let \( a_n \) denote the number of ways of tiling a \( 2 \times n \) board by \( 1 \times 2 \) rectangles, \( 2 \times 3 \) rectangles, and \( 2 \times 2 \) squares. These tiles can be placed horizontally or vertically. So \( a_1 = 1 \), \( a_2 = 3 \), and \( a_3 = 6 \). Find with proof a recursive formula for \( a_n \) for \( n \geq 4 \).