Homework 1

Due at lecture, Monday October 2

Instructions: Before starting this assignment, please read the section on "Homework" at `http://web.stanford.edu/class/math61cm/61cm-info.pdf`

1. Use mathematical induction to check that $\sum_{j=1}^{n} j2^j = (n-1)2^{n+1} + 2$ for $n = 1, 2, \ldots$.

   Note: The principle of mathematical induction is one of the basic properties of the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$, and we shall use it frequently; it says that if a subset $\mathcal{P} \subset \mathbb{N}$ has the properties (a) $1 \in \mathcal{P}$, and (b) $n \in \mathcal{P} \Rightarrow n+1 \in \mathcal{P}$ for each $n = 1, 2, \ldots$, then $\mathcal{P} = \mathbb{N}$. Slightly more concretely, that is the same as saying that if P_n is a (true or false) proposition for each $n = 1, 2, \ldots$ and if (a) P_1 is true, and (b) for each n we are able to check that P_{n+1} is true whenever P_n is true, then P_n <u>is</u> true for all $n = 1, 2, \ldots$.

2. Prove that $1 + \sqrt{2}$ is irrational (i.e. not rational).
   (Recall that x is rational means that $\exists$ integers p, q with $q \neq 0$ such that $x = p/q$.)

   Hint: First show that $1 + \sqrt{2}$ is rational if and only if $\sqrt{2}$ is rational. If $\sqrt{2}$ is rational, then we can write $\sqrt{2} = p/q$, where p, q are positive integers without common factors.

3. Prove or give a counterexample to the following:

   Claim: Let V be a vector space over a field F, and suppose that W_1, W_2 and W_3 are subspaces of V. Suppose also that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.

4. Give an example of a subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication but is not a subspace.

5. Let V be a vector space over a field F and U, W be subspaces of V. Suppose $U \cup W$ is also a subspace. Show that either $U \subseteq W$ or $W \subseteq U$.

6. Let a, b, c, d be elements of a field F, and consider the system of equations

   $$ax + by = 0,\ cx + dy = 0$$

   Show that $x = y = 0$ is the only solution if and only if $ad - bc \neq 0$.

7. Using the dot product, prove, for any vectors $\underline{x}, \underline{y} \in \mathbb{R}^n$:

   (a) The parallelogram law: $\|\underline{x} - \underline{y}\|^2 + \|\underline{x} + \underline{y}\|^2 = 2(\|\underline{x}\|^2 + \|\underline{y}\|^2)$

   (b) The law of cosines: $\|\underline{x} - \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 - 2\|\underline{x}\|\,\|\underline{y}\| \cos\theta$, assuming $\underline{x}, \underline{y}$ are non-zero and θ is the angle between $\underline{x}$ and $\underline{y}$ as discussed in lecture.

   (c) Give a geometric interpretion of identities (a) and (b).
   That is, describe what (a) is saying about the parallelogram determined by $\underline{x}, \underline{y}$—i.e. $OACB$ where $\overrightarrow{OA} = \underline{x}$, $\overrightarrow{OB} = \underline{y}$, $\overrightarrow{OC} = \underline{x} + \underline{y}$, and what (b) is saying about the triangle determined by $\underline{x}$ and $\underline{y}$—i.e. OAB, where $\overrightarrow{OA} = \underline{x}$, $\overrightarrow{OB} = \underline{y}$.

8. Give a bijective proof that $\binom{k}{r}\binom{n}{k} = \binom{n}{r}\binom{n-r}{k-r}$.

9. Let a_n denote the number of ways of tiling a $2 \times n$ board by 1×2 rectangles, 2×3 rectangles, and 2×2 squares. These tiles can be placed horizontally or vertically. So $a_1 = 1$, $a_2 = 3$, and $a_3 = 6$. Find with proof a recursive formula for a_n for $n \geq 4$.