Fall 2018

61DM Handout: Counting

Enumerative combinatorics refers to counting problems. We use $[n]$ to denote the set of positive integers from 1 to n.

A *permutation* of a set S is an ordering of the elements of S. Equivalently, if S has n elements, it is a map from $[n]$ to S such that no two elements of $[n]$ map to the same element of S. If $S = \{a, b, c\}$, then S has six permutations: *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*.

Example 1: Permutations. Let a_n denote the number of permutations of an n-element set S. Note that this number only depends on the size of the set S. It satisfies $a_0 = 1$ and $a_n = na_{n-1}$ for each positive integer n. Indeed, there are n ways to map the number n, and a_{n-1} ways to map the remaining elements. This an example of a *recursive* formula, which shows how to determine a later term in a sequence from earlier terms. We have $a_n = n!$, where $0! = 1$ and, for $n > 0$, $n! = 1 \cdot 2 \cdots n = \prod_{i=1}^{n} i$.

Example 2: Combinations. The number of ways of choosing a subset of k elements from an n element set is denoted by $\binom{n}{k}$. This satisfies $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Indeed, we can form each permutation by first picking out the first k elements. This can be done in $\binom{n}{k}$ ways. We can then order these k elements in $k!$ ways. To complete the permutation, we need to order the remaining $n - k$ elements, and this can be done in $(n-k)!$ ways. Thus, $n! = \binom{n}{k}k!(n-k)!$.

A map $f : A \to B$ is an *injection* (in other words, is one-to-one) if each element of B is mapped to by at most one element of A. For example, $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is not an injection since $g(-1) = g(1)$. However, if we restrict the domain of g to the nonnegative reals, then g is an injection. The map f is a *surjection* (in other words, is onto) if each element of the range gets mapped to. Note that the example g described above is not a surjection since no negative number gets mapped to by g. If we restrict the range of g to the nonnegative reals, then g becomes a surjection. A *bijection* is a map which is both an injection and a surjection. If we restrict both the domain and range of g to the nonnegative real numbers, then g is a bijection.

Two sets A and B have the same size if there is a bijection from A to B. For example $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ are of the same size because there is a way of matching up each element of A with an element of B given by the bijection $f : A \to B$ defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$.

Example 3: Symmetry of the binomial coefficients. We have the identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

One way to see this is to substitute into the formula established in Example 2 above:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

Another way is by a bijective proof. To do this, we need to find sets A and B with $|A| = \binom{n}{k}$ and $|B| = \binom{n}{n-k}$ and a bijection f from A to B. So let A be the set of k-element subsets of $[n]$

and B be the set of $(n-k)$-element subsets of $[n]$. Define the map $f : A \to B$ by mapping a set $S \subset A$ with $|S| = k$ to its complement $[n] \setminus S \in B$.

Double counting is another combinatorial technique. It is a way of proving an identity through counting the number of elements in a set in two different ways.

Example 4: Pascal's identity: For $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The left hand side counts the number of ways of choosing a k-element subset of $[n]$. The number n is either in the k-set or not in the k-set. If n is in the k-set, then there are $\binom{n-1}{k-1}$ ways to complete the k-set, by picking any $k-1$ elements from the remaining $n-1$ elements. If n is not in the k-set, then there are $\binom{n-1}{k}$ ways to complete the k-set, by picking any k elements from the remaining $n-1$ elements. An alternative proof of Pascal's identity is by plugging in the formula for $\binom{n}{k}$ and verifying both sides are the same.

Example 5: We have the identity

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The left hand side counts the number of ways of choosing n elements from $[2n]$. Note that, for each n-element subset S of $[2n]$, its intersection with $[n]$ has size k, for some $0 \leq k \leq n$. There are thus $\binom{n}{k}$ ways of choosing k elements to be in S from $[n]$. Having picked the intersection $S \cap [n]$, we have $|S \cap [2n] \setminus [n])| = n-k$. We could identity the k elements in $[2n] \setminus [n]$ that we choose not to include in S, and there are $\binom{n}{k}$ ways of doing this. These k elements in $[n]$ to include in S and the k elements in $[2n] \setminus [n]$ we choose to not include in S determine the n-element subset of $[2n]$. Thus $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Example 6: Counting subsets. Let t_n denote the number of subsets of $[n]$. For each subset $S \subset [n-1]$, we get two subsets of $[n]$ containing S, namely S and $S \cup \{n\}$. We thus have $t_n = 2t_{n-1}$. By induction using $t_0 = 1$, we get $t_n = 2^n$. This is an example of a linear homogeneous recurrence relation, where the n^{th} term in the sequence is a linear combination of a constant number of preceding terms.

Example 7: Counting tilings and the Fibonacci sequence. The Fibonacci sequence $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, \ldots$ satisfies the linear homogeneous recurrence relation $F_{n+2} = F_{n+1} + F_n$.

Let b_n denote the number of ways of tiling a $2 \times n$ board (consisting of two rows of n squares in each row) with 1×2 rectangles. For tiling, note that the 1×2 rectangle can be used either horizontally as a 1×2 rectangle or vertically as a 2×1 rectangle. We will prove $b_n = F_{n+1}$. Indeed, one can check this identity holds for $n = 0, 1, 2$. For larger n, observe that in a tiling either the top left square in the $2 \times n$ board is in a vertical 2×1 or in a horizontal 1×2. In the former case, there are b_{n-1} choices to complete the tiling. In the latter case, the 1×2 rectangle below the one in the top left corner must be filled by another horizontal 1×2, leaving a remaining $2 \times (n-2)$ board with b_{n-2} ways to fill it. This gives the recursive equation $b_n = b_{n-1} + b_{n-2}$. By induction, $b_n = F_{n+1}$.