Enumerative combinatorics refers to counting problems. We use \([n]\) to denote the set of positive integers from 1 to \(n\).

A permutation of a set \(S\) is an ordering of the elements of \(S\). Equivalently, if \(S\) has \(n\) elements, it is a map from \([n]\) to \(S\) such that no two elements of \([n]\) map to the same element of \(S\). If \(S = \{a, b, c\}\), then \(S\) has six permutations: \(abc, acb, bac, bca, cab,\) and \(cba\).

**Example 1:** Permutations. Let \(a_n\) denote the number of permutations of an \(n\)-element set \(S\). Note that this number only depends on the size of the set \(S\). It satisfies \(a_0 = 1\) and \(a_n = na_{n-1}\) for each positive integer \(n\). Indeed, there are \(n\) ways to map the number \(n\), and \(a_{n-1}\) ways to map the remaining elements. This an example of a recursive formula, which shows how to determine a later term in a sequence from earlier terms. We have \(a_n = n!\), where \(0! = 1\) and, for \(n > 0\), \(n! = 1 \cdot 2 \cdots n = \prod_{i=1}^{n} i\).

**Example 2:** Combinations. The number of ways of choosing a subset of \(k\) elements from an \(n\) element set is denoted by \(\binom{n}{k}\). This satisfies \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\). Indeed, we can form each permutation by first picking out the first \(k\) elements. This can be done in \(\binom{n}{k}\) ways. We can then order these \(k\) elements in \(k!\) ways. To complete the permutation, we need to order the remaining \(n-k\) elements, and this can be done in \((n-k)!\) ways. Thus, \(n! = \binom{n}{k} k!(n-k)!\).

A map \(f : A \to B\) is an injection (in other words, is one-to-one) if each element of \(A\) maps to at most one element of \(B\). For example, \(g : \mathbb{R} \to \mathbb{R}\) given by \(g(x) = x^2\) is not an injection since \(g(-1) = g(1)\). However, if we restrict the domain of \(g\) to the nonnegative reals, then \(g\) is an injection. The map \(f\) is a surjection (in other words, is onto) if each element of the range gets mapped to. Note that the example \(g\) described above is not a surjection since no negative number gets mapped to by \(g\). If we restrict the range of \(g\) to the nonnegative reals, then \(g\) becomes a surjection. A bijection is a map which is both an injection and a surjection. If we restrict both the domain and range of \(g\) to the nonnegative real numbers, then \(g\) is a bijection.

Two sets \(A\) and \(B\) have the same size if there is a bijection from \(A\) to \(B\). For example \(A = \{a, b, c\}\) and \(B = \{1, 2, 3\}\) are of the same size because there is a way of matching up each element of \(A\) with an element of \(B\) given by the bijection \(f : A \to B\) defined by \(f(a) = 1, f(b) = 2,\) and \(f(c) = 3\).

**Example 3:** Symmetry of the binomial coefficients. We have the identity

\[
\binom{n}{k} = \binom{n}{n-k}.
\]

One way to see this is to substitute into the formula established in Example 2 above:

\[
\binom{n}{n-k} = \frac{n!}{(n-k)!((n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.
\]
Another way is by a bijective proof. To do this, we need to find sets $A$ and $B$ with $|A| = \binom{n}{k}$ and $|B| = \binom{n}{n-k}$ and a bijection $f$ from $A$ to $B$. So let $A$ be the set of $k$-element subsets of $[n]$ and $B$ be the set of $(n - k)$-element subsets of $[n]$. Define the map $f : A \rightarrow B$ by mapping a set $S \subseteq A$ with $|S| = k$ to its complement $[n] \setminus S \in B$.

Double counting is another combinatorial technique. It is a way of proving an identity through counting the number of elements in a set in two different ways.

**Example 4:** Pascal’s identity: For $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$  

The left hand side counts the number of ways of choosing a $k$-element subset of $[n]$. The number $n$ is either in the $k$-set or not in the $k$-set. If $n$ is in the $k$-set, then there are $\binom{n-1}{k-1}$ ways to complete the $k$-set, by picking any $k - 1$ elements from the remaining $n - 1$ elements. If $n$ is not in the $k$-set, then there are $\binom{n-1}{k}$ ways to complete the $k$-set, by picking any $k$ elements from the remaining $n - 1$ elements. An alternative proof of Pascal’s identity is by plugging in the formula for $\binom{n}{k}$ and verifying both sides are the same.

**Example 5:** We have the identity

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$  

The left hand side counts the number of ways of choosing $n$ elements from $[2n]$. Note that, for each $n$-element subset $S$ of $[2n]$, its intersection with $[n]$ has size $k$, for some $0 \leq k \leq n$. There are thus $\binom{n}{k}$ ways of choosing $k$ elements to be in $S$ from $[n]$. Having picked the intersection $S \cap [n]$, we have $|S \cap [2n] \setminus [n]| = n - k$. We could identity the $k$ elements in $[2n] \setminus [n]$ that we choose not to include in $S$, and there are $\binom{n}{k}$ ways of doing this. These $k$ elements in $[n]$ to include in $S$ and the $k$ elements in $[2n] \setminus [n]$ we choose to not include in $S$ determine the $n$-element subset of $[2n]$. Thus $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

**Example 6:** Counting subsets. Let $t_n$ denote the number of subsets of $[n]$. For each subset $S \subseteq [n-1]$, we get two subsets of $[n]$ containing $S$, namely $S$ and $S \cup \{n\}$. We thus have $t_n = 2t_{n-1}$. By induction using $t_0 = 1$, we get $t_n = 2^n$. This is an example of a linear homogeneous recurrence relation, where the $n^{th}$ term in the sequence is a linear combination of a constant number of preceding terms.

**Example 7:** Counting tilings and the Fibonacci sequence. The Fibonacci sequence $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, \ldots$ satisfies the linear homogeneous recurrence relation $F_{n+2} = F_{n+1} + F_n$.

Let $b_n$ denote the number of ways of tiling a $2 \times n$ board (consisting of two rows of $n$ squares in each row) with $1 \times 2$ rectangles. For tiling, note that the $1 \times 2$ rectangle can be used either horizontally as a $1 \times 2$ rectangle or vertically as a $2 \times 1$ rectangle. We will prove $b_n = F_{n+1}$. Indeed, one can check this identity holds for $n = 0, 1, 2$. For larger $n$, observe that in a tiling either the top left square in the $2 \times n$ board is in a vertical $2 \times 1$ or in a horizontal $1 \times 2$. In the former case, there are $b_{n-1}$ choices to complete the tiling. In the latter case, the $1 \times 2$ rectangle below the one in the top left corner must be filled by another horizontal $1 \times 2$, leaving a remaining $2 \times (n - 2)$ board with $b_{n-2}$ ways to fill it. This gives the recursive equation $b_n = b_{n-1} + b_{n-2}$. By induction, $b_n = F_{n+1}$.