Mathematics Department Stanford University  
Math 61DM Practice Final Examination v.1, December 12, 2016  

3 Hours  

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.  

Total score is capped at 35 points; number of available points is 37.  

If necessary, continue solutions on backs of pages  
Note: work sheets are provided for your convenience, but will not be graded  

<table>
<thead>
<tr>
<th>Q.1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.2</td>
<td></td>
</tr>
<tr>
<td>Q.3</td>
<td></td>
</tr>
<tr>
<td>Q.4</td>
<td></td>
</tr>
<tr>
<td>Q.5</td>
<td></td>
</tr>
<tr>
<td>Q.6</td>
<td></td>
</tr>
<tr>
<td>T/35</td>
<td></td>
</tr>
</tbody>
</table>

Name (Print Clearly):  ____________________________  

I understand and accept the provisions of the honor code (Signed) ____________________________
1(a) (3 points) Consider the following $5 \times 5$ matrix

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

(i) What is the determinant of $M$? (ii) What is the rank of $M$?
Explain how you came to your answer.

(b) (3 points): Find the set of all solutions of the inhomogeneous system $Ax = y$ where

$$A = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
2 & 1 & 1 & 3 & 2 \\
1 & 1 & 2 & 0 & 3 \\
0 & 0 & 1 & -1 & 1
\end{pmatrix}, \quad y = \begin{pmatrix}
1 \\
4 \\
1 \\
-1
\end{pmatrix}$$

(Give your answer as an affine space.)
2(a) (5 points) Suppose $H_1, H_2, \ldots, H_m$ are hyperplanes that do not cover all of $\{0,1\}^n$. Prove that there are at least $2^{n-m}$ points of $\{0,1\}^n$ left uncovered.

(b) (2 points): Give a construction showing that the bound in part (a) is tight.
3(a) **(3 points.)** Suppose a friend claims that there is a positive integer $n$ and a subset $A \subset \mathbb{F}_3^n$ which takes up at least 70% of $\mathbb{F}_3^n$ and contains no affine line (three collinear points) in $\mathbb{F}_3^n$. Give a complete proof that your friend is wrong.

3(b) **(4 points.)** (i) For sufficiently large integers $n$, is there a subset of $\mathbb{F}_3^n$ of size $2.9^n$ with no affine line? (You may cite a result proved in class) (ii) Show that for all positive integers $n$, there is a subset of $\mathbb{F}_3^n$ of size $2^n$ with no affine line.
4(a) (3 points): Find all eigenvalues and corresponding eigenvectors for the matrix

\[
\begin{pmatrix}
5 & -2 & 1 \\
-2 & 2 & 2 \\
1 & 2 & 5
\end{pmatrix}
\]

(b) (2 points): Suppose a bipartite graph $G$ on 10 vertices is 4-regular. (i) What is the largest eigenvalue for $G$? (ii) What is the smallest eigenvalue for $G$?
5(a) (4 points): (i) What does the spectral theorem say? (ii) Why is it useful in graph theory?

(c) (3 points): Suppose an (undirected, simple) graph on $n$ vertices has largest eigenvalue $n - 1$. Prove that the graph is the complete graph $K_n$. 
6(a) (2½ points): Suppose $V$ is a finite dimensional real vector space, and $e_1, \ldots, e_n$ is a basis for $V$. Show that the linear maps $f_i : V \to \mathbb{R}$, $i = 1, \ldots, n$, defined by $f_i(\sum_{j=1}^n a_j e_j) = a_i$ give a basis (called the dual basis) of $V^* = \mathcal{L}(V, \mathbb{R})$.

(b) (2½ points): Suppose that $A \in \mathcal{L}(V, V)$ is linear, $V$, etc., as above. Show that trace $A = \sum_{j=1}^n f_j(Ae_j)$ is independent of the choice of the basis of $V$, and if $A$ is symmetric, then trace $A$ is the sum of the eigenvalues of $A$, counted with multiplicity, i.e. if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues, then trace $A = \sum_{j=1}^k \lambda_j \dim N(A - \lambda_j I)$.

Note: $f_j(Ae_j)$ is the $jj$ entry of the matrix of $A$ in the basis $e_1, \ldots, e_n$, so the trace is the sum of the diagonal entries of the matrix.