Mathematics Department Stanford University
Math 61DM Practice Final Examination v.2, December 12, 2016

3 Hours

Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

39 total points, score will be out of 35 (you may view 3(b) as extra credit)

<table>
<thead>
<tr>
<th>Q.1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.2</td>
<td></td>
</tr>
<tr>
<td>Q.3</td>
<td></td>
</tr>
<tr>
<td>Q.4</td>
<td></td>
</tr>
<tr>
<td>Q.5</td>
<td></td>
</tr>
<tr>
<td>T/35</td>
<td></td>
</tr>
</tbody>
</table>

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________
1(a) (3 points): Find all eigenvalues and corresponding eigenvectors for the matrix
\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \]

1(b) (3 points): Find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by the vectors
\[ v_1 = (1, 1, 0, 0)^T, v_2 = (0, 1, 1, 0)^T, v_3 = (0, 0, 1, 1)^T. \]
Let $A$ be an $n \times n$ matrix. Let $f(\lambda) = \det(A - \lambda I)$.

2(a) (2 points): Prove that $f(\lambda)$ is a polynomial of degree $n$ in $\lambda$.

2(b) (2 points): What is the coefficient of $\lambda^n$ in $f(\lambda)$?

2(c) (2 points): What is the coefficient of $\lambda^{n-1}$ in $f(\lambda)$?

2(d) (2 points) Prove that $\text{Tr}(A) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. 
3(a) (3 points.) What is the statement of the finite field Kakeya theorem?

3(b) (4 points.) What is the main idea of the proof?
4(a) (4 points): Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix. The matrix $A$ is called positive definite if for every nonzero column vector $x \in \mathbb{R}^n$, we have $x^T Ax$ is positive. Prove that $A$ is positive definite $\iff$ all the eigenvalues of $A$ are positive.

Hint: Spectral Theorem.

(b) (3 points): Can the adjacency matrix of a graph without loops be positive definite? Justify your answer.
5(a) (3 points): Suppose $G$ is a graph with an ordering of its vertices $v_1, \ldots, v_n$ such that each vertex $v_i$ is adjacent to at most $d$ vertices $v_j$ with $j < i$. Prove that the chromatic number of $G$ is at most $d + 1$.

(b) (4 points): Prove that for every graph $G$ on $n$ vertices, $G$ or its complement has chromatic number at least $\sqrt{n}$. 