# 61DM Handout: Generating Functions

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Suppose we have a sequence $a_0, a_1, a_2, \dots$ of numbers. The generating function for this sequence is defined as

$$G(x) = \sum_{i=0}^{\infty} a_i x^i.$$

We will not worry here about issues of convergence. Let us see a few examples. If $a_i = 1$ for all $i$, then $G(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$. This can be seen by multiplying each side by $1-x$, and noticing that the sum telescopes. If $a_i = a^i$ for all $i$, then $G(x) = \frac{1}{1-ax}$. This follows from substituting in $ax$ for $x$ in the previous formula. One can see that if $F$ is the generating function for $\{a_i\}$ and $G$ is the generating function $\{b_i\}$, then $F+G$ is the generating function for $\{a_i + b_i\}$. It is also helpful to sometimes multiply generating functions, or take their derivative or integral.

We exhibit the usefulness of generating functions through an application determining an explicit formula for Fibonacci numbers. Recall that these numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The generating function for the Fibonacci sequence is $G(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots$. For $n \geq 2$, we substitute in the identity $F_n = F_{n-1} + F_{n-2}$ and obtain

$$G(x) = F_0 + F_1 x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n = F_0 + F_1 x + \sum_{n \geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n$$

We have

$$\sum_{n\geq 2} F_{n-1} x^n = x \sum_{n \geq 1} F_n x^n = x\left(G(x) - F_0\right) = xG(x) - F_0 x.$$

and

$$\sum_{n\geq 2} F_{n-2} x^n = x^2 G(x).$$

Substituting, we have

$$G(x) = F_0 + F_1 x + xG(x) - F_0 x + x^2 G(x).$$

Solving for $G(x)$:

$$G(x) = \frac{F_0 + (F_1 - F_0)x}{1 - x - x^2} = \frac{x}{1-x-x^2} = \frac{a}{1-\phi x} + \frac{b}{1 - \tau x},$$

where the last equation is by taking the partial fraction decomposition, using $\phi = \frac{1+\sqrt{5}}{2}$ and $\tau = \frac{1-\sqrt{5}}{2}$ are the roots of the equation $y^2 - y - 1 = 0$, and $a, b$ are some real numbers for which this equation will be satisfied. We get $a+b=0$ and $-b\phi - a\tau = 1$. Solving for $a$ and $b$, we obtain $a = \frac{1}{\sqrt{5}}$ and $b = \frac{1}{\sqrt{5}}$. Hence,

$$G(x) = \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\tau x} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^n - \tau^n)x^n,$$

so that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$