Mathematics Department Stanford University
Math 61DM Homework 2
DUE AT TA SECTION, FRIDAY OCT. 6

1. Let V be the vector space of polynomials with coefficients in a field F of degree at most 5. Let U be the subspace of V consisting of polynomials of the form $a(z^2 + z^5) + bz$ with $a, b \in F$. Find a subspace W such that every element $v \in V$ can be written in one and only one way as the sum of an element in U and another element in W.

2. Prove that there exists a quadratic polynomial $ax^2 + bx + c$ whose graph passes through the three points $(6, 1)$, $(2, 0), (1, 7)$. Is it unique?

3. Find a single homogeneous linear equation with unknowns x_1, x_2, x_3 such that the solution set is the span of the 2 vectors $(2, 1, -3)^{\mathrm{T}}, (1, -1, 0)^{\mathrm{T}}$.

4. Let V be a vector space and suppose $S = \{v_1, \ldots, v_k\}$ is a finite set of linearly dependent vectors in V. Prove that there is a proper subset of S whose span is equal to the span of S. By a proper subset, we mean a subset $T \subset S$ such that $T \neq S$.

5. Use Gaussian elimination (in $\mathbb{R}$) to show that the solution set of the homogeneous system

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 0 \\ 2x_1 + 3x_2 - 2x_3 - 6x_4 &= 0 \\ 2x_1 - 2x_2 - 2x_3 + 4x_4 &= 0 \\ 3x_1 - x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

is a plane through $\underline{0}$ (i.e. the span of 2 l.i. vectors), and find 2 l.i. vectors whose span is the solution space.

6. In parts (a) and (b) we will show that multiplicative inverses exist in $\mathbb{Z}/p\mathbb{Z}$. So, you should not use this fact in your answers to (a) and (b). However, feel free to use other basic facts about arithmetic in $\mathbb{Z}/p\mathbb{Z}$.

(a) Suppose p is prime and $a \in \mathbb{Z}/p\mathbb{Z}$, $a \neq 0$. For any $x, y \in \mathbb{Z}/p\mathbb{Z}$ show that if $ax = ay$ (with multiplication in $\mathbb{Z}/p\mathbb{Z}$, i.e. modulo p) then $x = y$. [Hint: you will need to unwrap the definition of "mod p multiplication", and make use of the hypotheses that p is prime and $a \neq 0$.]

(b) By considering the set $\{a0, a1, \ldots, a(p-1)\}$ over $\mathbb{Z}/p\mathbb{Z}$, or otherwise, show that there exists $b \in \mathbb{Z}/p\mathbb{Z}$ such that $ab = 1$ (again, with multiplication mod p).

(c) In the set of integers $\mathbb{Z}$, solve the system of equations

$$2x + y = 2 \mod 5,\ 3x - 2y = 0 \mod 5,$$

by using Gaussian elimination in $\mathbb{Z}/(5\mathbb{Z})$.

7. Find an explicit formula for a_n in each case below using generating functions:

(a) $a_0 = 2$, $a_1 = 1$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n > 1$.

(b) $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n > 1$.

(**Hint for (b):** Once you determine the generating function, you can takes its integral to determine the coefficients of the generating function.)

8. There are two rabbits. At the end of each year, the rabbit population has doubled, but then one rabbit dies. Let a_n be the number of rabbits after n years, so $a_0 = 2$, $a_1 = 3$, and $a_2 = 5$.

(a) Find a recurrence equation for a_n.

(b) Solve for a_n explicitly.

9. Use the inclusion-exclusion principle to determine how many numbers up to 2017 are not divisible by 2, 3, or 7.