1. Suppose $V$ is a finite dimensional vector space over a field $F$, $e_1, \ldots, e_n$ is a basis for $V$, and let $f_1, \ldots, f_n$ be the dual basis of $V^* = \mathcal{L}(V,F)$ (i.e. $f_i(e_j) = 1$ if $j = i$, 0 otherwise), and suppose that $A \in \mathcal{L}(V,V)$. Show that trace $A = \sum_{j=1}^n f_j(\langle Ae_j \rangle)$ is independent of the choice of the basis $e_1, \ldots, e_n$ of $V$.

Note: In HW#5 Q2(i) you showed that the trace is well-defined in inner product spaces; this is now extended to arbitrary fields and no inner products. Notice that $f$ is multilinear but not homogeneous (as the first term is of degree 2), the polynomial $f$ is both multilinear and homogeneous. If each of its terms are of degree 2, then there is a degree 2 polynomial that vanishes on $E$, i.e., $f(x) = 0$ for all $x \in E$.

2. (a) If $S$ is the 2 dimensional subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 1, 0, 0)^T, (0, 0, 1, 1)^T$, find an orthonormal basis for $S$ and find the matrix of the orthogonal projection of $\mathbb{R}^4$ onto $S$.

(b) If $S$ is the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 0, 1)^T, (1, 1, 0, 0)^T, (0, 0, 1, 1)^T$, find an orthonormal basis for $S$, and find the matrix of the orthogonal projection of $\mathbb{R}^4$ onto $S$.

3. Showing all row operations, calculate the determinant of

$$
\begin{pmatrix}
10 & 11 & 12 & 13 & 426 \\
100 & 101 & 101 & 102 & 2000 \\
\end{pmatrix}.
$$

4. This problem asks you to prove the factor theorem.

(i) Suppose $f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$ is a degree $d$ polynomial (so $a_d \neq 0$) with coefficients in a field $F$. Prove that the number of zeros (elements $x \in F$ with $f(x) = 0$) is at most $d$.

(ii) Conversely, if $E \subset F$ with $|E| \leq d$, then there is a degree $d$ polynomial that vanishes on $E$, i.e., $f(x) = 0$ for all $x \in E$.

5. Suppose $f(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$ is a degree $d$ polynomial with coefficients in $\mathbb{R}$. Prove that $f(x)$ has a nonzero multiple in which all the exponents are prime numbers.

For instance, such a multiple of $f(x) = x^2 - x + 5$ is $x^5 + 4x^3 + 5x^2 = (x^2 + x)(x^2 - x + 5)$. You may assume that that there are infinitely many prime numbers.

6. Determine with proof the dimension over a field $F$ of the space of multilinear homogeneous polynomials of degree $d$ in $n$ variables.

A polynomial in $n$ variables is homogeneous of degree $d$ if each of its terms are of degree $d$. A polynomial in $n$ variables is multilinear if it is linear in each variable. For example, if $n = 3$ and the variables are $x_1, x_2, x_3$, then the polynomial $x_1x_2x_3 + x_1x_3$ is multilinear but not homogeneous (as the first term is of degree three and the second is of degree 2), the polynomial $x_1x_2 + x_3^2$ is homogeneous (of degree 2) but not multilinear (since it is quadratic in $x_3$), and $x_1x_2 + x_2x_3$ is both multilinear and homogeneous.

7. Prove that if all distances between $m$ distinct points in $\mathbb{R}^n$ are equal, then $m \leq n + 1$.

8. Suppose $A$ and $B$ are subsets of $\mathbb{R}^3$ such that the distances between all members of $A$ and all members of $B$ are equal. Assume $|A| \leq |B|$. Explain why $|A| \leq 2$. Your explanation may be geometric in nature.