1. Using two separate methods:
   (i) by computing $\det A$ and $\text{adj } A$, and
   (ii) using elementary row operations to reduce $(A|I)$ to $(I|B)$,
   calculate the inverse of the $3 \times 3$ matrix
   \[
   \begin{pmatrix}
   1 & 4 & 3 \\
   1 & 4 & 5 \\
   2 & 5 & 1 \\
   \end{pmatrix}
   \]

2. Let
   \[
   \Delta = \det \begin{pmatrix}
   1 & 1 & \cdots & 1 \\
   x_1 & x_2 & \cdots & x_n \\
   \vdots & \vdots & \cdots & \vdots \\
   x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\
   \end{pmatrix},
   \]
   which is called a Vandermonde determinant. Show that
   \[
   \Delta = \prod_{1 \leq i < j \leq n} (x_j - x_i).
   \]
   (Hint: Consider using row operations to reduce the $n$-variable case to the $(n-1)$-variable case.)

3. Suppose $A$, $B$ are $n \times n$ matrices and that $AB = I$ (so $B$ is a “right inverse” of $A$; we do not assume $BA = I$ so we are not given that $B$ is the full inverse $A^{-1}$).
   (i) Prove that $\det A \neq 0$.
   (ii) Prove that $B = (\det A)^{-1} \text{adj } A$, i.e. that $B$ is necessarily is the unique inverse $(\det A)^{-1} \text{adj } A$ of $A$ and so $BA = I$ after all.
   (iii) Give an example of an $n \times m$ matrix $C$ and an $m \times n$ matrix $D$ such that $CD = I_n$ but $DC \neq I_m$.

4. Let $A_1, \ldots, A_m$ be distinct subsets of $[n]$. Assume that their pairwise symmetric differences have only two sizes (so there are positive integers $a$ and $b$ such that $|A_i \Delta A_j| = a$ or $b$ for all $1 \leq i < j \leq m$). Prove that $m \leq 1 + \frac{n(n+1)}{2}$.
   Give an example of such a family of size $m = 1 + \frac{n(n-1)}{2}$, for each $n$.
   [The symmetric difference $A \Delta B$ of two sets $A$, $B$ is the set $(A \setminus B) \cup (B \setminus A)$ of elements in $A$ or $B$, but not both.]

5. Let $S$ be a subset of $F_3^n$ and suppose that for every pair of distinct vectors $u, v \in S$ there is an index $i$, $1 \leq i \leq n$, for which $v_i \equiv u_i + 1 \pmod{3}$. Show that $|S| \leq 2^n$.
   [Here $u_i$ is the $i$-th coordinate of $u$, and $v_i$ is the $i$-th coordinate of $v$. Also, the above property (that $v_i \equiv u_i + 1 \pmod{3}$ for some $i$) holds for all $2\binom{|S|}{2}$ ordered pairs $(u, v)$ with $u, v \in S$ distinct.]

6. A spherical 2-distance set $S$ is a collection of points in $\mathbb{R}^n$ such that each has distance one from the origin and the pairwise distances of points in $S$ take on only two values (i.e. $\|x\| = 1$ for each $x \in S$ and there exist positive real numbers $a, b \in \mathbb{R}$ such that $\|x - y\| = a$ or $b$ for each distinct pair $x, y \in S$).
   Show that $|S| \leq n(n + 3)/2$.
   [You can refer to M15 in explaining your answer.]
   (Remark: There are beautiful geometric examples showing this bound is tight for $n = 2, 6, 22$.)