Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages

Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
Q.5
T/30

Name (Print Clearly):

I understand and accept the provisions of the honor code (Signed)
Let □ denote the $1 \times 1$ square and $\Gamma$ be the $2 \times 2$ square with one of the four $1 \times 1$ squares removed.

1 (a) (2 points): Let $f(n)$ be the number of ways to tile a $2 \times n$ rectangle by $\Gamma$. Determine and prove an explicit formula for $f(n)$.

1(b) (4 points): Let $g(n)$ be the number of ways to tile a $2 \times n$ rectangle by $\Gamma$ and □.
(i) Determine a recursive formula for $g(n)$.
(iii) Explain why there are constants $a, b, c$ such that $g(n) = a\alpha^n + b\beta^n + c\gamma^n$ for all $n$, where $\alpha, \beta, \gamma$ are the solutions to $x^3 - x^2 - 4x - 2 = 0$. 
2(a) (5 points): (i) State the rank nullity theorem. (ii)-(iii): Suppose $A$ is an $m \times n$ matrix over a field $\mathbb{F}$ and $C(A) = \mathbb{F}^m$. (ii) Show that $m \leq n$. (iii) If in addition $Ax = b$ has a unique solution for every $b \in \mathbb{F}^m$, show that $m = n$. 

2(b) (4 points):
Suppose $V$ is a vector space over a field $\mathbb{F}$, $v_1, \ldots, v_k \in V$ and $V = \text{span}\{v_1, \ldots, v_k\}$. Show that there is a sub-collection $\{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$, $i_1 < i_2 < \ldots < i_l$ (possibly $l = 0$), such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\}$ is a basis for $V$.

Hint for (b): Analogously to the proof of the basis theorem, consider a minimal size subcollection that spans $V$, or a maximal size subcollection which is linearly independent.
3(a) (2 points) What does the oddtown theorem say? Why is it true?

3(b) (3 points) Suppose town with $n$ citizens has $m$ clubs $C_1, \ldots, C_m$ of one type, and $m$ clubs $D_1, \ldots, D_m$ of another type such that $C_i$ and $D_i$ have an odd number of members in common for each $i$, and if $i > j$, then clubs $C_i$ and $D_j$ have an even number of common members. Prove that $m \leq n$.

Hint for (b): Consider each club as a vector. If $A$ is the matrix whose $i$th row is the vector corresponding to $C_i$, and $B$ is the matrix whose $j$th column is the vector corresponding to $D_j$, then consider the rank of the matrix product $AB$. 
4 (5 points): Using Gaussian elimination, (i) find a basis for the null space $N(A)$, (ii) the dimension of the column space of $A$, (iii) find a basis for the column space of $A$ if

$$A = \begin{pmatrix}
1 & 2 & 3 & 1 & 1 \\
-1 & -2 & -2 & 0 & 1 \\
0 & 1 & 0 & 0 & 2
\end{pmatrix}$$
Let $G$ be a graph, $A(G)$ be the adjacency matrix of $G$, and $\ell$ be a positive integer.

5(a) (3 points): Give a combinatorial description for the sum of the nondiagonal entries of $A(G)^\ell$.

5(b) (2 points): Prove that if a graph $G$ has no isolated vertices and $\ell$ is even, then every entry of $A(G)^\ell$ on the diagonal is positive.