Unless otherwise indicated, you can use results covered in lecture or homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

Q.1
Q.2
Q.3
Q.4
T/25

Name (Print Clearly): ____________________________

I understand and accept the provisions of the honor code (Signed) ____________________________
1(a) (4 points): Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 2, 0)^T$, $(1, 0, 0, 3)^T$, $(0, 2, 0, 1)^T$, and write down an explicit formula (involving numbers and matrix operations only) for the matrix of the orthogonal projection to this subspace (but you do not need to compute it).

1(b) (3 points) Suppose $A$ is an $m \times n$ real matrix. Prove that the dimensions of $C(A)$ and $C(A^T)$ are the same.
2 (5 points): Let $a, b, c, n$ be positive integers and $B_1, B_2, \ldots, B_m$ be subsets of $[n]$ such that for each pair $1 \leq i \neq j \leq m$, we have $|B_i \cap B_j| = a$, $|B_i \Delta B_j| = b$, or $|B_i| = |B_j| + c$. Prove that $m = O(n^3)$. 
3(a) (4 points.) Let $n \geq 4$ and $S_1, \ldots, S_m$ be distinct subsets of $[n]$ such that, for each $1 \leq i < j \leq m$, we have $|S_i \cap S_j| \in \{0, 1, n-2, n-1\}$. Prove that $m \leq \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} = \frac{n^2 + 3n + 4}{2}$.

Hint: You may want to consider the large sets (of size at least $n-1$) differently from the small sets (of size less than $n-1$).

3(b) (2 points.) Show that this bound is tight for $n = 4$, and get a lower bound of $\binom{n}{0} + \binom{n}{1} + \binom{n}{2}$ for all $n$. 
4(a) (4 points). Prove that if $A$ is an $n \times n$ matrix with each entry equal to an integer, and if $\det A = 1$, then each entry of $A^{-1}$ is also an integer.

4(b) (3 points): Find the determinant and the inverse of the matrix

$$A = \begin{pmatrix} 3 & 6 & -9 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$