Summary of definitions and main results

Definitions we’ve covered: $R$–algebra, $k^{th}$ tensor power $T^k(M)$, tensor algebra $T^*(M)$, $k^{th}$ symmetric power $\text{Sym}^k(M)$, symmetric algebra $\text{Sym}^*(M)$, exterior power $\wedge^k M$, exterior algebra $\bigwedge^* M$, group ring, (linear) representation, degree of a representation, faithful representation, trivial representation, permutation representation, regular representation, homomorphism and isomorphism of representations, $G$-equivariant map, intertwiner, minimal polynomial of a linear map.

Main results: using right exactness to compute tensor products, construction & universal properties for tensor, symmetric, and exterior powers and algebras, equivalent definitions of a group representation.

Warm-Up Questions

1. (a) We defined how to form the tensor product $M \otimes_R N$ of a right $R$–module $M$ and a left $R$–module $N$. What would go wrong with this construction if $M$ instead had the structure of a left $R$–module?
(b) Show that if $M$ is an $(S,R)$–bimodule and $N$ a left $R$–module, the tensor product $M \otimes_R N$ has the structure of an $S$–module. Why must the left action of $S$ and the right action of $R$ on $M$ commute?

2. Verify that the tensor product of maps respects composition:

$$(\hat{\phi} \otimes \hat{\psi}) \circ (\phi \otimes \psi) = (\hat{\phi} \circ \phi) \otimes (\hat{\psi} \circ \psi).$$

3. Let $\phi : M \to M'$ be a map of right $R$–modules, and $\psi : N \to N'$ be a map of left $R$–modules.

(a) Show by example that even if $\phi$ and $\psi$ both inject, their tensor product $\phi \otimes \psi$ may not be injective.
(b) Show that if $\phi$ and $\psi$ are both surjective, then their tensor product $\phi \otimes \psi$ will be surjective.
(c) Show that if $\phi$ and $\psi$ are both isomorphisms, then their tensor product $\phi \otimes \psi$ will be an isomorphism.

Hint: Isomorphisms have inverses. Use Warm-Up Problem 2.

4. Fill in the details of the proof of that the tensor product associates (Dummit–Foote 10.4 Theorem 14).

5. Let $M$ be a right $R$–module and $N_1, \ldots, N_n$ a set of left $R$–modules. Verify that the tensor product distributes over direct sums (Dummit–Foote 10.4 Theorem 17). There is a unique group isomorphism

$$M \otimes_R (N_1 \oplus \cdots \oplus N_n) \cong (M \otimes_R N_1) \oplus \cdots \oplus (M \otimes_R N_n).$$

Conclude that if $N$ is a left $R$–module, $R^n \otimes_R N \cong N^n$.

6. Show that the following alternate definition of an $R$–algebra $A$ is equivalent to the one from class. Given a commutative ring $R$, an $R$–algebra $A$ is an $R$–module $A$ with a ring structure such that the multiplication map $A \times A \to A$ is $R$–bilinear.

7. Let $R$ be a commutative ring, and $M$ and $R$–module.

(a) Verify that, if $R$ does not have characteristic 2, then the submodule

$$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k \mid m_i = m_j \text{ for some } i \neq j \rangle \subseteq T^k M$$

defining the exterior power $\bigwedge^k M$ is equal to the submodule

$$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k - \text{sign}(\sigma)m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)} \mid \sigma \in S_k \rangle.$$

(b) Are these submodules the same when $R$ has characteristic 2?
8. Let $R$ be a commutative ring and $M$ and $R$–module. Verify the universal properties for the $R$–modules

(a) $T^k(M)$  
(b) $\text{Sym}^k(M)$  
(c) $\bigwedge^k(M)$

and for $R$–algebras

(d) $T^*(M)$  
(e) $\text{Sym}^*(M)$

9. Let $G$ be a group and $V$ an $\mathbb{F}$–vector space. Show that the following are all equivalent ways to define a (linear) representation of $G$ on $V$.

i. A group homomorphism $G \to \text{GL}(V)$.

ii. A group action (by linear maps) of $G$ on $V$.

iii. An $\mathbb{F}[G]$–module structure on $V$.

10. Let $R$ be a commutative ring. Show that the group ring $R[\mathbb{Z}] \cong R[t, t^{-1}]$. Show that $R[\mathbb{Z}/n\mathbb{Z}] \cong R[t]/(t^n - 1)$. What is the group ring $R[\mathbb{Z}^n]$? The group ring $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$?

11. Let $\phi : G \to \text{GL}(V)$ be any group representation. What is the image of the identity element in $\text{GL}(V)$?

12. Compute the sum and product of $(1 + 3e_{(12)} + 4e_{(123)})$ and $(4 + 2e_{(12)} + 4e_{(13)})$ in the group ring $\mathbb{Q}[S_3]$.

13. Let $G$ be a group and $R$ a commutative ring. Show that $R[G]$ is commutative if and only if $G$ is abelian.

14. Given any representation $\phi : G \to \text{GL}(V)$, prove that $\phi$ defines a faithful representation of $G/\ker(\phi)$.

15. (a) Find an explicit isomorphism $T$ between the following two representations of $S_2$.

$$S_2 \to \text{GL}(\mathbb{R}^2)$$

$$(1 \ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) Prove that the following two representations of $S_2$ are not isomorphic.

$$S_2 \to \text{GL}(\mathbb{R}^2)$$

$$(1 \ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) Give a geometric description of the action and the bases for $\mathbb{R}^2$ associated to each matrix group.

16. Fix an integer $n > 0$. Recall the following example from class: The symmetric group $S_n$ acts on $\mathbb{C}^n$ by permuting a basis $e_1, e_2, \ldots, e_n$. We saw that this representation has two subrepresentations,

$$D = \text{span}_\mathbb{C}(e_1 + e_2 + \cdots + e_n)$$

and

$$U = \{ a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots + a_n = 0 \}.$$

Show that, as a $\mathbb{C}S_n$–module, $\mathbb{C}^n$ is the direct sum $\mathbb{C}^n \cong D \oplus U$.

17. Let $A$, $B$, $C$ be linear maps $V \to V$, with $C$ invertible. Verify the following properties of the trace.

(a) $\text{Trace}(CAC^{-1}) = \text{Trace}(A)$ (so trace does not depend on choice of basis or matrix representing $A$).

(b) $\text{Trace}(cA + B) = c\text{Trace}(A) + \text{Trace}(B)$ for any scalar $c$.

(c) $\text{Trace}(AB) = \text{Trace}(BA)$ but $\text{Trace}(AB) \neq \text{Trace}(A)\text{Trace}(B)$ in general.

(d) $\text{Trace}(A) = \text{Trace}(A^T)$.

(e) $\text{Trace}(\text{Id}_V) = \dim(V)$.

(f) $\text{Trace}(A)$ is the sum of the eigenvalues of $A$ (with algebraic multiplicity).

(g) If $A$ has characteristic polynomial $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\text{Trace}(A) = a_{n-1}$.

(h) If $V = U \oplus W$ and $U$, $W$ are stabilized by $A$, then $\text{Trace}(A) = \text{Trace}(A|_U) + \text{Trace}(A|_W)$. 


Assignment Questions

1. (a) Use the right-exactness of the functor $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} -$ and the short exact sequence of $\mathbb{Z}$-modules
   \[ 0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \]
   to (re)compute the abelian group $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.
   
   (b) Let $k$ be a field and let $R = k[x,y]$. Using any method you prefer, give simple descriptions of the following $R$-modules, and determine their dimensions over $k$.
   \[ \frac{R}{(x)} \otimes_R \frac{R}{(x-y)}, \quad \frac{R}{(x)} \otimes_R \frac{R}{(x-1)}, \quad \frac{R}{(y-1)} \otimes_R \frac{R}{(x-y)} \]

2. Let $R$ be a commutative ring and $M$ and $R$-module.
   
   (a) For any commutative ring $R$ and $R$-module $M$, show that the $R$-module $T^* M := \bigoplus_{i=0}^{\infty} M^\otimes i$ has the structure of an $R$-algebra. Verify that this algebra may not be commutative.
   
   (b) A similar proof shows that $\text{Sym}^* M := \bigoplus_{i=0}^{\infty} \text{Sym}^i(M)$ and $\wedge^* M := \bigoplus_{i=0}^{\infty} \wedge^i M$ are $R$-algebras. You do not need to give a full proof, but verify that multiplication is well-defined for these spaces (it is independent of representative of an equivalence class of elements in these quotients).

3. Let $F$ be a field of characteristic zero and $V$ a vector space over $F$ with basis $\{x_1, \ldots, x_n\}$.
   
   (a) Verify that $\text{Sym}^k(V)$ is a vector space over $F$ with basis given by the set of monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ of total degree $k$. (Remark: There are $\binom{n+k-1}{k}$ such monomials).
   
   Hint: To show these elements are linearly independent, it is enough to use the universal property to define a symmetric multilinear map $V^k \to F$ that factors through $\text{Sym}^k V$ which takes value 1 on one basis element and 0 on all others.
   
   (b) Verify that $\wedge^k V$ is isomorphic to the $F$–vector space with a basis given by elements of the form $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. (Remark: There are $\binom{n}{k}$ such elements).
   
   (c) Suppose that $A : V \to V$ is a diagonalizable linear map with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with multiplicity). Compute the eigenvalues of the maps induced by $A$ on $T^k V$, $\text{Sym}^k(V)$, and $\wedge^k V$.
   
   (d) Show that you can identify $\text{Sym}^* V$, and $\wedge^* V$ as direct summands of $T^* V$ via the (split) maps
   \[ x_1 x_2 \cdots x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \quad \text{and} \quad x_1 \wedge x_2 \wedge \cdots \wedge x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma) \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \]
   (We are using the assumption that $F$ has characteristic zero, so the integer $k!$ is invertible in $F$.)
   
   (e) Show that $V \otimes_F V \cong \text{Sym}^2(V) \oplus \wedge^2 V$.
   
   (f) Show that if $V$ has dimension at least 2, then $V \otimes_F V \cong \text{Sym}^3(V) \oplus \wedge^3 V$.

4. Let $G$ be a finite group, and $F$ a field. You may use properties of the trace without proof.
   
   (a) Let $G \to GL(U)$ be any representation of $G$. Citing facts from linear algebra (which you don’t need to prove), explain why the trace of the matrix representing a given element $g \in G$ is well-defined in the sense that it will be the same in any isomorphic representation of $G$.
   
   (b) A permutation representation of $G$ on a finite-dimensional $F$-vector space $V$ is a linear representation $\rho : G \to GL(V)$ in which elements act by permuting some basis $B = \{b_1, \ldots, b_m\}$. Show that, with respect to the basis $\{b_1, \ldots, b_m\}$, for each element $g \in G$, $\rho(g)$ is represented by an $m \times m$ permutation matrix, a square matrix that has exactly one entry 1 in each row and each column, and zero elsewhere. Use this description of matrices $\rho(g)$ to show that the trace of $\rho(g)$ is equal to the number of basis elements $b_i$ fixed by $\rho(g)$. 

(c) Our first example of a permutation representation was given by the action of $S_n$ on $F^n$ by permuting the basis $e_1, \ldots, e_n$. Show, in contrast, that the subrepresentation

$$U = \{ a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots + a_n = 0 \} \subseteq F^n$$

is not a permutation representation with respect to any basis for $U$.

*Hint:* Warm-up Question 17(h). What is the trace of an $n$–cycle?

(d) The group ring of $F[G]$ is a left module over itself. This corresponds to permutation representation of the group $G$ on the underlying vector space $F[G]$, called the (left) regular representation of $G$. Find the degree of this representation. In what basis is this a permutation representation, and how many $G$–orbits does this basis have?

(e) For any $g \in G$, compute the trace of the matrix representing $g$ in the regular representation.

5. Let $V$ be a $C[x]$–module that is finite dimensional over $C$, where $x$ acts on $V$ by a $C$–linear map $T$. According to the structure theorem for finitely generated modules over a PID, we can write

$$V \cong \frac{C[x]}{(p_1(x))} \oplus \frac{C[x]}{(p_2(x))} \oplus \cdots \oplus \frac{C[x]}{(p_k(x))}$$

for some monic polynomials $p_i(x) \in C[x]$ such that $p_1(x)$ divides $p_2(x)$, $p_2(x)$ divides $p_3(x)$, etc.

The monic polynomial $p_k(x)$ is called the *minimal polynomial* of $T$, and the product $p_1(x)p_2(x) \cdots p_k(x)$ is called the *characteristic polynomial* of $T$. By construction the minimal and characteristic polynomials have the same set of roots (possibly with different multiplicities).

(a) Verify that if $\lambda \in C$ is a root of $p_i(x)$, then $\frac{p_i(x)}{(x - \lambda)} \in \frac{C[x]}{(p_i(x))}$ is an eigenvector of $T$ with eigenvalue $\lambda$.

(b) Suppose that $\mu \in C$ is not a root of $p_k(x)$ (and therefore not a root of $p_i(x)$ for any $i$). Show that $\mu$ is not an eigenvalue of $T$. Conclude that the eigenvalues of $T$ are precisely the roots of the minimal polynomial $p_k(x)$.

*Hint:* Recall that an eigenvector for $\mu$ is a nonzero element of $\ker(T - \mu I)$, where $I$ is the identity matrix. Consider the projection of a $\mu$–eigenspace onto the summand $\frac{C[x]}{(p_i(x))}$ for each $i$, and notice that the polynomial $(x - \mu)$ is coprime to $p_i(x)$.

(c) Suppose the roots of $p_k(x)$ are distinct, ie, they each occur with multiplicity one. Show that $T$ is diagonalizable. *Hint:* Chinese Remainder Theorem.

(d) Show that $\text{Ann}(V) = (p_k(x))$.

(e) Show that $\text{Ann}(V)$ is equal to the set

$$\{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in C[x] \mid a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I \text{ is the zero map} \}$$

Conclude that if $p(T) = 0$ for some polynomial $p(x)$, every eigenvalue of $T$ is a root of $p(x)$.

(f) Suppose the linear map $T$ has finite order, that is, $T^n = I$ for some $n \in \mathbb{Z}_{\geq 0}$. Show that $T$ is diagonalizable, and that all its eigenvalues are $n$th roots of unity.

(g) Let $G$ be a finite group of order $n$, and let $\rho : G \to GL(V)$ be a representations of $G$ on a finite dimensional vector space $V$. Conclude that for every $g \in G$ the linear map $\rho(g)$ is diagonalizable, and its eigenvalues are all $n$th roots of unity.

6. **Bonus (Optional).** Let $C^d$ be the canonical permutation representation of the symmetric group $S_d$, and consider the induced action on $\Lambda^k C^d$. Prove that

$$\frac{1}{d!} \sum_{\sigma \in S_d} \left( \text{Trace} (\sigma \circ \Lambda^k C^d) \right)^2 = 2 \quad \text{for any } d \geq 1 \text{ and } 0 \leq k \leq d - 1.$$

We will see that with *character theory*, this result implies that $\Lambda^k U$ is an irreducible $S_d$–representation for all $0 \leq k \leq d - 1$. 