General Equilibrium

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> "From the time of Adam Smith's *Wealth of Nations* in 1776, one recurrent theme of economic analysis has been the remarkable degree of coherence among the vast numbers of individual and seemingly separate decisions about the buying and selling of commodities. In everyday, normal experience, there is something of a balance between the amounts of goods and services that some individuals want to supply and the amounts that other, differerent individuals want to sell [sic]. Would-be buyers ordinarily count correctly on being able to carry out their intentions, and would-be sellers do not ordinarily find themselves producing great amounts of goods that they cannot sell. This experience of balance is indeed so widespread that it raises no intellectual disquiet among laymen; they take it so much for granted that they are not disposed to understand the mechanism by which it occurs."
>
> Kenneth Arrow (1973)

1 Introduction

General equilibrium analysis addresses precisely how these "vast numbers of individual and seemingly separate decisions" referred to by Arrow aggregate in a way that coordinates productive effort, balances supply and demand, and leads to an efficient allocation of goods and services in the economy. The answer economists have provided, beginning with Adam Smith and continuing through to Jevons and

*Various sections of these notes draw heavily on lecture notes written by Felix Kubler; some of the other sections draw on Mas-Colell, Whinston and Green.

Walras is that it is the price system plays the crucial coordinating and equilibrating role: the fact the everyone in the economy faces the same prices is what generates the common information needed to coordinate disparate individual decisions.

You doubtless are familiar with the standard treatment of equilibrium in a single market. Price plays the role of equilibrating demand and supply so that all buyers who want to buy at the going price can, and do, and similarly all sellers who want to sell at the going price also can and do, with no excess or shortages on either side. The extension from this *partial equilibrium* in a single market to *general equilibrium* reflects the idea that it may not be legitimate to speak of equilibrium with respect to a single commodity when supply and demand in that market depend on the prices of other goods. On this view, a coherent theory of the price system and the coordination of economic activity has to consider the simultaneous general equilibrium of all markets in the economy. This of course raises the questions of *(i)* whether such a general equilibrium exists; and *(ii)* what are its properties.

A recurring theme in general equilibrium analysis, and economic theory more generally, has been the idea that the competitive price mechanism leads to outcomes that are efficient in a way that outcomes under other systems such as planned economies are not. The relevant notion of efficiency was formalized and tied to competitive equilibrium by Vilfredo Pareto (1909) and Abram Bergson (1938). This line of inquiry culminates in the Welfare Theorems of Arrow (1951) and Debreu (1951). These theorems state that there is in essence an equivalence between Pareto efficient outcomes and competitive price equilibria.

Our goal in the next few lectures is to do some small justice to the main ideas of general equilibrium. We'll start with the basic concepts and definitions, the welfare theorems, and the efficiency properties of equilibrium. We'll then provide a proof that a general equilibrium exists under certain conditions. From there, we'll investigate a few important ideas about general equilibrium: whether equilibrium is unique, how prices might adjust to their equilibrium levels and whether these levels are stable, and the extent to which equilibria can be characterized and changes in exogenous preferences or endowments will have predictable consequences. Finally we'll discuss how one can incorporate production into the model and then time and uncertainty, leading to a brief discussion of financial markets.

2 The Walrasian Model

We're going to focus initially on a pure exchange economy. An *exchange economy* is an economy without production. There are a finite number of agents and a finite number of commodities. Each agent is endowed with a bundle of commodities. Shortly the world will end and everyone will consume their commodities, but before this happens there will be an opportunity for trade at some set prices. We want to know whether there exist prices such that when everyone tries to trade their desired amounts at these prices, demand will just equal supply, and also what the resulting outcome will look like — whether it will be efficient in a well-defined sense and how it will depend on preferences and endowments.

2.1 The Model

Consider an economy with I agents $i \in \mathcal{I} = \{1, ..., I\}$ and L commodities $l \in \mathcal{L} = \{1, ..., L\}$. A bundle of commodities is a vector $x \in \mathbb{R}^L_+$. Each agent i has an endowment $e^i \in \mathbb{R}^L_+$ and a utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$. These endowments and utilities are the primitives of the exchange economy, so we write $\mathcal{E} = ((u^i, e^i)_{i \in \mathcal{I}})$.

Agents are assumed to take as given the market prices for the goods. We won't have much to say about where these prices come from, although we'll say a bit later on. The vector of market prices is $p \in \mathbb{R}^L_+$; all prices are nonnegative.

Each agent chooses consumption to maximize her utility given her budget constraint. Therefore, agent i solves:

$$\max_{x \in \mathbb{R}^L_+} u^i(x) \qquad \text{s.t. } p \cdot x \leq p \cdot e^i.$$

The budget constraint is slightly different than in standard price theory. Recall that the familiar budget constraint is $p \cdot x \leq w$, where w is the consumer's initial wealth. Here the consumer's "wealth" is $p \cdot e^i$, the amount she could get if she sold her entire endowment. We can write the budget set as

$$\mathcal{B}^i(p) = \{x : p \cdot x \leq p \cdot e^i\}.$$

We'll occasionally use this notation below.

2.2 Walrasian Equilibrium

We now define a *Walrasian equilibrium* for the exchange economy. A Walrasian equilibrium is a vector of prices, and a consumption bundle for each agent, such that *(i)* every agent's consumption maximizes her utility given prices, and *(ii)* markets clear: the total demand for each commodity just equals the aggregate endowment.

Definition 1 *A* ***Walrasian equilibrium*** *for the economy $\mathcal{E}$ is a vector $(p, (x^i)_{i \in \mathcal{I}})$ such that:*

1. *Agents are maximizing their utilities: for all $i \in \mathcal{I}$,*

$$x^i \in \arg\max_{x \in \mathcal{B}^i(p)} u^i(x)$$

2. *Markets clear: for all $l \in \mathcal{L}$,*

$$\sum_{i \in \mathcal{I}} x_l^i = \sum_{i \in \mathcal{I}} e_l^i.$$

2.3 Pareto Optimality

The second important idea is the notion of Pareto optimality, due to the Italian economist Vilfredo Pareto. This notion doesn't have anything to do with equilibrium per se (although we'll see the close connection soon). Rather it considers the set of feasible allocations and identifies those allocations at which no consumer could be made better off without another being made worse off.

Definition 2 *An allocation $(x^i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{I \cdot L}$ is* ***feasible*** *if for all $l \in \mathcal{L}$: $\sum_{i \in \mathcal{I}} x_l^i \leq \sum_{i \in \mathcal{I}} e_l^i$.*

Definition 3 *Given an economy $\mathcal{E}$, a feasible allocation x is* ***Pareto optimal*** *(or* ***Pareto efficient****) if there is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some $i \in \mathcal{I}$.*

You should note that Pareto efficiency, while it has significant content, says essentially nothing about distributional justice or equity. For instance, it can be Pareto efficient for one guy to have everything and everyone else have nothing. Pareto efficiency just says that there aren't any "win-win" changes around; it's quiet on how social trade-offs should be resolved.

2.4 Assumptions

As we go along, we're going to repeatedly invoke a bunch of assumptions about consumers' preferences and endowments. We summarize the main ones here.

(A1) For all agents $i \in \mathcal{I}$, u^i is continuous.

(A2) For all agents $i \in \mathcal{I}$, u^i is increasing, i.e. $u^i(x') > u^i(x)$ whenever $x' \gg x$.

(A3) For all agents $i \in \mathcal{I}$, u^i is concave.

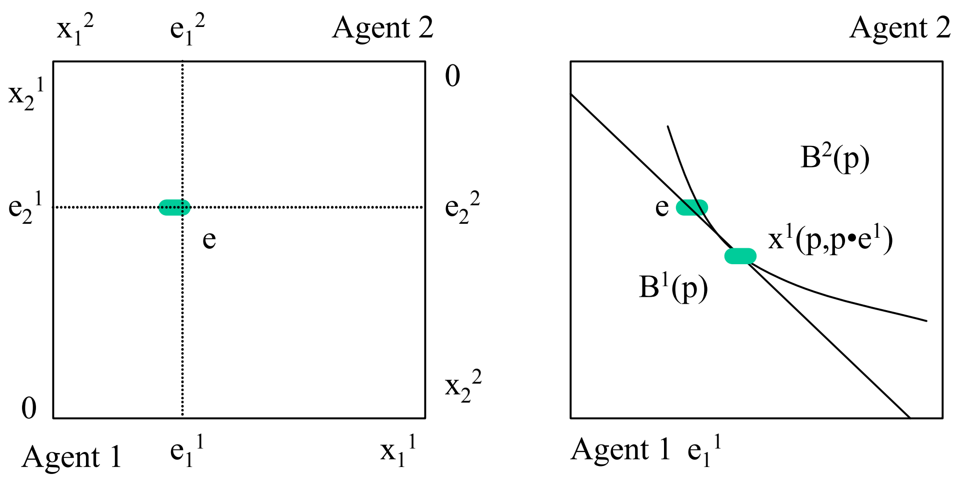
(A4) For all agents $i \in \mathcal{I}$, $e^i \gg 0$.

The first three assumptions — continuity, monotonicity and concavity of the utility function — should be familiar from consumer theory. Some of these are a bit stronger than necessary (e.g. monotonicity can be weakened to local nonsatiation, concavity to quasi-concavity), but we're not aiming for maximum generality. The last assumption, about endowments, is new and is a big one. It says that everyone has a little bit of everything. This turns out to be important and you'll see where it comes into play later on.

3 A Graphical Example

General equilibrium theory can quickly get into the higher realms of mathematical economics. Nevertheless a lot of the big ideas can be expressed in a simple two-person two-good exchange economy. A useful graphical way to study such economies is the Edgeworth box, after F. Edgeworth, a famous Cambridge (U.K.) economist of the 19th century.[1]

Figure 1(a) presents an Edgeworth box. The bottom left corner is the origin for agent 1. The bottom line is the x-axis for Agent 1 and the left side is the y-axis. In the picture, agent 1's endowment is $e^1 = (e^1_1, e^1_2)$. For agent 2, the origin is the top right corner and everything is flipped upside down and backward. Every point in the box represents a (non-wasteful) allocation of the two goods.



Figures 1(a) and 1(b): The Edgeworth Box

Figure 1(b) adds prices into the picture. Given prices p_1, p_2 for the two goods, the budget line for agent 1 is the line with slope p_1/p_2 through the endowment point e. This is also the budget line for agent 2. So this line divides the Edgeworth box into the two budget sets $B^1(p)$ and $B^2(p)$. Each agent will then choose consumption to maximize utility given prices. In Figure 1(b), agent 1's Marshallian demand $x^1(p, p \cdot e)$ is represented by the familiar tangency condition.

[1]Apparently the name is something of a misnomer, as it seems that Edgeworth boxes were first drawn by Pareto — or so I read on the internet.

As we change prices, the Marshallian demands of the two agents will also change. Note that what matters, of course, is the *relative prices* of the two goods, as these determine the slope of the budget line. Figure 2 traces out the Marshallian demand of agent 1 as we vary the relative prices. The dotted line is called agent 1's *offer curve*.

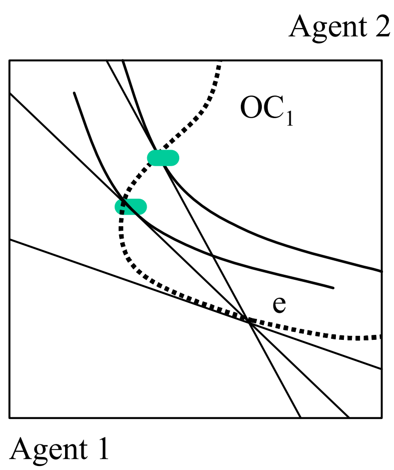
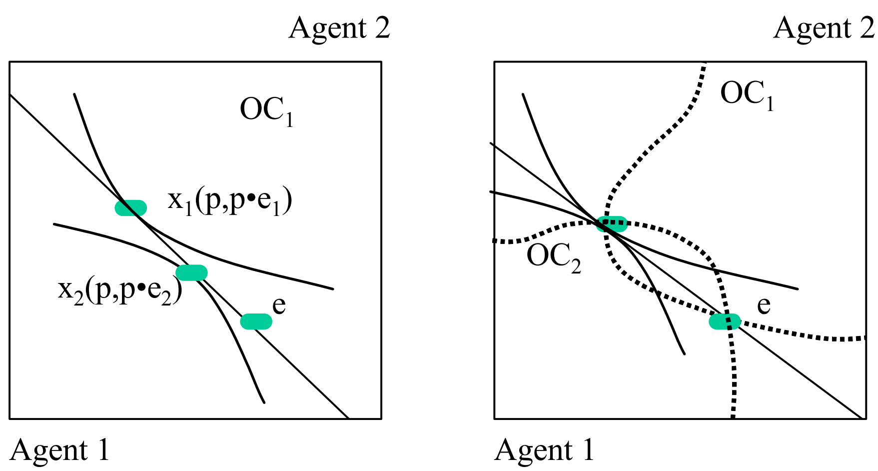


Figure 2: Offer Curve for Agent 1

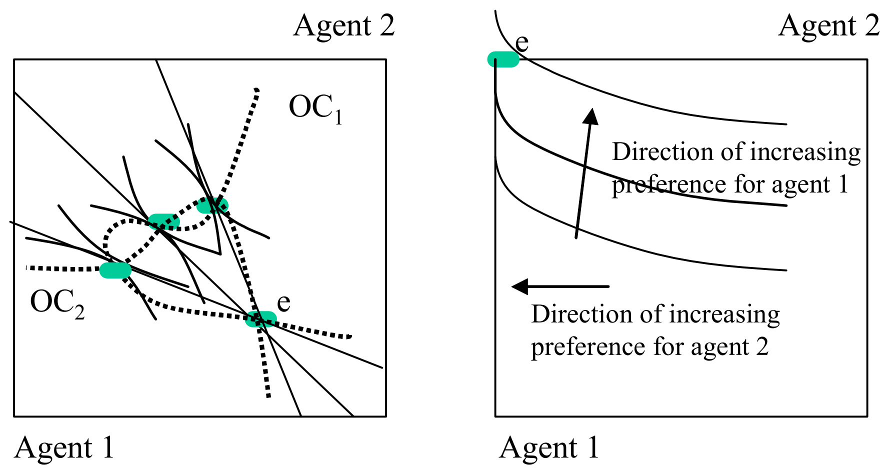
Walrasian equilibrium requires that both agents consume their Marshallian demands given prices and also that these demands are compatible. So what we want to do is set relative prices, find the Marshallian demands of the two agents, and see whether or not demand equals supply in the two markets. Figure 3(a) represents a situation where prices do not simultaneously clear the two markets. In this picture, at the given prices, agent 2 is willing to supply some amount of good 2, but less than agent 1 wants to consume. So good 2 is in excess demand. In contrast, agent 1 is willing to supply more of good 1 than agent 2 demands. So good 2 is in excess supply.

In Figure 3(b), prices do clear the market and we have a Walrasian equilibrium at the point x. In equilibrium, starting from the endowment point e, agent 1 sells good 1 to buy good 2; agent 2 does the reverse. The crucial point is that both markets clear. Note that the Walrasian equilibrium allocation is the *intersection of the two offer curves.* That the point x lies on the offer curve of agent i means that x it represents the Marshallian demand of that agent given prices p and endowment e. That the point x is the intersection of the two offer curves means that at the given prices, demands are compatible and markets clear. These are conditions (1) and (2) in the definition of Walrasian equilibrium.



Figures 3(a) and 3(b): Dis-equilibrium and Equilibrium in the Edgeworth Box

Two natural questions to ask about Walrasian equilibrium are *(i)* is it unique? and *(ii)* does it always exist? Both questions have negative answers. Figure 4(a) presents an example with multiple Walrasian equilibria (we're revisit this example later). In the figure, given the endowment e, the offers curves of the two agents intersect three times. So there are three Walrasian equilibria.



Figures 4(a) and (b): Non-uniqueness and Non-existence of Equilibrium

Figure 4(b) presents a different example where Walrasian equilibrium does not exist. In this example, Agent 2 starts with all of good 1 and this is the only good she cares about. Agent 1 starts with all of good 2 and none of good 1. He cares about both goods, but the slope of his indifference curve when he has none of good 1 is infinite. That is, he has infinite marginal utility for his very first unit of good 1. In this example, for any prices p, agent 2 will insist on consuming her endowment — that is, all of good 1. Moreover, there are no prices p at which agent 1 would not insist on buying at least a little bit of good 1. Therefore for any prices p good 1 will always be in excess demand and there cannot be a Walrasian equilibrium. Note that this example violates assumption (A4), which requires that the endowment be an interior point in the Edgeworth box.

It is also possible to use the Edgeworth box to depict the idea of Pareto optimality. This is done in Figure 5. The *Pareto set* in this picture is the set of all allocations such that to make one agent better off would require making the other agent worse off. Figure 5 also shows the *contract curve*. This is the part of the Pareto set that both agents prefer to the endowment e. It seems natural to expect that if the agents were to start at their endowments and strike a mutually agreeable bargain, they would reach a point on the contract curve assuming that bargaining does not leave mutual gains from trade on the table.

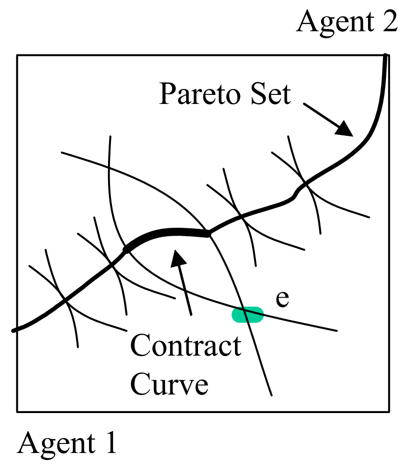


Figure 5: The Contract Curve

Figure 5 also provides some intuition for a key result in general equilibrium theory: *any Walrasian equilibrium is Pareto optimal* (or lies on the Pareto set). The reason is as follows. At a Walrasian equilibrium, the budget line will separate the two "as good as" sets of the agents (as we saw in Figure 3(b)). Thus, there will be no alternative to the Walrasian outcome that would make both agents better off. Therefore any Walrasian equilibrium is Pareto optimal. The Pareto set, of course, is the set of *all* Pareto optimal allocations, so an alternative statement is that any Walrasian equilibrium allocation lies on the Pareto set. This result is known as the *first theorem of welfare economics*.

4 The Welfare Theorems

We now turn to a more formal statement of the theorem suggested above — that every Walrasian equilibrium allocation is a Pareto optimal allocation. We then prove a converse result that if an initial allocation is Pareto optimal, there is a Walrasian equilibrium at which no trade occurs.

Theorem 1 (First Welfare Theorem) *(Arrow, 1951; Debreu, 1951) Let* $(p, (x^i)_{i\in\mathcal{I}})$ *be a Walrasian equilibrium for the economy* $\mathcal{E}$*. Then if (A2) holds, the allocation* $(x^i)_{i\in\mathcal{I}}$ *is Pareto optimal.*

Proof. By way of contradiction, suppose there is a feasible allocation $(\hat{x}^i)_{i\in\mathcal{I}}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i'. By revealed preference and (A2), $p \cdot \hat{x}^i \geq p \cdot x^i$ for all $i \in \mathcal{I}$, and also $p \cdot \hat{x}^{i'} > p \cdot x^{i'}$ (this is Walras' law). Therefore, because prices are non-negative, there must be at least one good l for which $\sum_i \hat{x}^i_l > \sum_i x^i_l = \sum_i e^i_l$. Therefore $\hat{x}$ is not feasible. *Q.E.D.*

This result provides formal support for Adam Smith's claim that individuals acting in their own interests end up behaving in a way that is efficient from a societal standpoint. It is a powerful statement about the efficiency properties of decentralized markets: despite the fact that there is no explicit social coordination and agents simply maximize their utilities given prices, the resulting equilibrium outcome is efficient from a social perspective.

Note that in a sense, the assumptions are quite weak. Given our model of the exchange economy, the only assumption on preferences that we require is monotonicity (and local nonsatiation would suffice). Of course, it should be emphasized that the model itself contains a large number of heroic implicit assumptions that seem highly unlikely to be satisfied in any real economy. Among these are: (1) all agents face precisely the same prices; (2) all agents are price takers — i.e. they take prices as given and don't believe that their purchasing decisions will move prices; (3) markets exist for all goods and agents can freely participate. Moreover, we have said nothing so far about how a group of agents might arrive at equilibium prices. So you'll probably want to withhold judgment on the efficiency of decentralized markets.

The first welfare theorem states that equilibrium outcomes are efficient. Our next result states that efficient outcomes are Walrasian equilibria given the correct prices and endowments.

Theorem 2 (Second Welfare Theorem) *(Arrow, 1951; Debreu, 1951) Let* $\mathcal{E}$ *be an economy that satisfies (A1)–(A4). If* $(e^i)_{i\in\mathcal{I}}$ *is Pareto optimal then there exists a price vector* $p \in \mathbb{R}^L_+$ *such that* $(p, (e^i)_{i\in\mathcal{I}})$ *is a Walrasian equilibrium for* $\mathcal{E}$*.*

Proof. To prove this we need a version of the separating hyperplane theorem. Suppose you have an open convex set $A \subset \mathbb{R}^n$ and a point $x \notin A$. Then there exists a $p \neq 0$ such that $p \cdot a \geq p \cdot x$ for all $a \in cl(A)$.

To prove the theorem, let's define:

$$A^i = \{a \in \mathbb{R}^L : e^i + a \geq 0 \text{ and } u^i(e^i + a) > u^i(e^i)\}.$$

Because u^i is concave, A^i is a convex set. Therefore the set

$$A = \sum_{i \in \mathcal{I}} A^i = \{a \in \mathbb{R}^L : \exists a_1 \in A_1, ..., a_I \in A_I \text{ with } a = \sum_{i \in \mathcal{I}} a_i\}$$

is a convex set. Moreover, $0 \notin A$, because if it were there would exist some $(a^i)_{i \in \mathcal{I}}$ with $\sum_{i \in \mathcal{I}} a^i = 0$ and $u^i(e^i + a^i) > u^i(e^i)$ for all $i \in \mathcal{I}$, contradicting the assumption that e is Pareto optimal.

The separating hyperplane theorem now implies that there is some price vector $p \neq 0$ such that $p \cdot a \geq 0$ for all $a \in cl(A)$. Furthermore, $p \geq 0$ because if $a \gg 0$ then $a \in A$ by monotonicity and if some $p_l < 0$ we could take a_l arbitrarily big and all the other $a_{l'}$ small but positive and get a contradiction. Because $p \neq 0$ and $p \geq 0$, this means that $p > 0$.

The claim is that this p will support the allocation e as a Walrasian equilibrium. Obviously e satisfies the market clearing part of the definition of equilibrium. Moreover, fixing prices p, consider a given agent i. Suppose $x^i \in \mathbb{R}^L_+$ and $u^i(x^i) > u^i(e^i)$. We will show that x^i is not in i's budget set, thus proving the individual optimization part of equilibrium. First, by definition of A^i and p, $p \cdot x^i \geq p \cdot e^i$. Moreover, by continuity, the fact that $u^i(x^i) > u^i(e^i)$ implies that for λ just less than 1, $u^i(\lambda x^i) > u^i(e^i)$. Therefore $p \cdot \lambda x^i \geq p \cdot e^i$. This can't be the case if $p \cdot x^i = p \cdot e^i$, so therefore $p \cdot x^i > p \cdot e^i$. *Q.E.D.*

Note that the second welfare theorem does not say that starting from a given endowment, every Pareto optimal allocation is a Walrasian equilibrium. Rather it says that if we were to start from a given endowment then for any Pareto optimal allocation there is a way to re-distribute resources and a set of prices that makes the allocation a Walrasian equilibrium outcome.

In practice, this means that decentralizing a Pareto optimal allocation is not simply a matter of identifying and specifying the correct prices (not that this would necessarily be easy). Large-scale re-distribution may be required as well. This limits the practical applicability of the theorem. Still, it is a useful result conceptually and for modelling. For instance, in complicated macroeconomic dynamic models, it can sometimes be hard to directly establish the existence of an equilibrium; in some cases, one can proceed by identifying Pareto optimal allocations and then showing a version of the second welfare theorem saying that the Pareto allocation can be supported as a Walrasian equilibrium.

Finally, one technical point. Observe that unlike with the first welfare theorem, convexity plays a crucial role in establishing the second welfare theorem. Indeed, at a formal level, the theorem is a direct application of the separating hyperplane theorem, where the equilibrium price vector separates the Pareto allocation e from the set of allocations preferred to e by at least one agent.

5 Characterizing Equilibrium

In this section, which follows MWG, chapter 16, we make a bunch of assumptions about utility functions being differentiable and concave and then use first order conditions to characterize Pareto optimal allocations. The idea is to give some intuition for what conditions must be satisfied on the margin at any Pareto optimal allocation, and hence, by the first Welfare theorem, at any Walrasian equilibrium. We also tie the set of Pareto optimal allocations to the set of allocations that maximize linear Bergson-Samuelson social welfare functions.

One way to identify the set of Pareto optimal allocations $x = (x^1, ..., x^I)$ is as solutions to the following program:

$$\begin{array}{lll} & \max_x \quad u^1(x_1^1, ..., x_L^1) & \\ \text{s.t.} & u^i(x_1^i, ..., x_L^i) \geq \overline{u}^i & \text{for } i = 2, ..., I \\ & \sum_i x_l^i \leq \sum_i e_l^i & \text{for } l = 1, ..., L. \end{array}$$

The idea here is to maximize the utility of the first consumer subject to feasibility and to the other consumers getting at least some pre-specified level of utility. By varying the level of required utility for consumers $2, ..., I$, we can recover the full set of Pareto optimal allocations.

Under assumptions (A1)–(A3), all of the constraints must be binding at the solution (if the utility constraint for i were slack we could reduce x^i be ε in all directions and increase x^1 by the same amount; if the resource constraint were slack we could increase either x^1 or one of the x^is). If was assume in addition that each agent has a differentiable utility function, the problem satisfies the conditions of the Kuhn-Tucker theorem, so we can use the Kuhn-Tucker conditions to characterize the solution.

Let λ^i denote the Lagrange multiplier on agent i's constraint and let μ_l denote the constraint on commodity l. The Kuhn-Tucker conditions are then:

$$\lambda^i \frac{\partial u^i}{\partial x_l^i} - \mu_l \leq 0, \qquad x_l^i \geq 0, \qquad \left(\lambda^i \frac{\partial u^i}{\partial x_l^i} - \mu_l\right) x_l^i = 0, \tag{1}$$

coupled with the requirement that each of the $(I-1)+L$ constraints is binding:

$$\begin{aligned} u^i(x_1^i, ..., x_L^i) &= \overline{u}^i \qquad \text{for } i = 2, ..., I \\ \sum_i x_l^i &= \sum_i e_l^i \qquad \text{for } l = 1, ..., L. \end{aligned}$$

In the first line, I've adopted the convention that $\lambda^1 = 1$; you'll see where this bit of notation comes in useful later. Note that because each of the constraints binds at the optimum, $\lambda^i > 0$ for $i = 2, ..., I$ and $\mu_l > 0$ for all l.

The Kuhn-Tucker conditions given in (1) are easy to interpret. Recall that λ^i is precisely the marginal value, or shadow price, of consumer i's income in terms of consumer 1's utility. That is, at the optimum taking a util away from agent i would allow us to increase agent 1's utility by λ^i. At the same time, μ_l is the shadow price on commodity l (again in terms of agent 1's utility). An extra unit of commodity l would allow us to increase agent 1's utility by μ_l while holding everyone else's utility constant.

Assuming that each consumer consumes a positive amount of each good at the optimum, so that $x_l^i > 0$ for all i, l, we can easily derive that at any Pareto efficient allocation, we have the following relationship:

$$MRS_{kl}^i = \frac{\partial u^i / \partial x_k^i}{\partial u^i / \partial x_l^i} = \frac{\partial u^j / \partial x_k^j}{\partial u^j / \partial x_l^j} = MRS_{kl}^j = \frac{\mu_k}{\mu_l}.$$

That is, at the optimum, the marginal rates of substitution of every agent for every commodity pair k, l must be equal to each other and to the ratio of the shadow prices μ_k and μ_l. This is precisely the tangency condition from our earlier Edgeworth box picture.

Within this simple framework of differentiable concave utility functions, we can link the Pareto optimal allocations to the set of Walrasian equilibria quite easily. Suppose that x is a Pareto optimal allocation as characterized above. Let $e^i = x^i$ and define prices $p_l = \mu_l$. Given these prices and endowments, consider the optimization problem facing consumer i :

$$\max_{\tilde{x}^i} u^i(\tilde{x}^i)$$
$$\text{s.t. } p \cdot \tilde{x}^i \leq p \cdot e^i$$

Again, we know the budget constraint will bind at the optimum given our assumptions (that's Walras' Law). Moreover, we can use the Kuhn-Tucker conditions to characterize the optimum. Letting $\nu^1, ..., \nu^I$ denote the Lagrange multipliers on the budget constraints of agents $1, ..., I$, the Kuhn-Tucker conditions state that a necessary and sufficient condition for $(x^1, ..., x^I; \nu^1, ..., \nu^I)$ to solve the I utility maximization problems given prices p is that for all i, l:

$$\frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l \leq 0 \qquad x_l^i \geq 0, \qquad \left(\frac{\partial u^i}{\partial x_l^i} - \nu^i \cdot p_l \right) \cdot x_l^i = 0, \tag{2}$$

and in addition, each of the resource constraints bind.

It's quite easy to see that if x is a Pareto optimal allocation, one solution is for each agent i to consume $x^i = (x_1^i, ..., x_L^i)$ with Lagrange multipliers $\nu^i = 1/\lambda^i$. Why? Because given prices $p_l = \mu_l$ and endowments $e^i = x^i$, there is an exact equivalence between the Kuhn-Tucker conditions of the I utility maximization problems and the Kuhn-Tucker conditions of the earlier Pareto problem.

Therefore it follows that if x is a Pareto optimal allocation, and $\mu_1, ..., \mu_L$ the commodity shadow prices from the Pareto problem above, then (μ, x) is a Walrasian equilibrium of the economy $\mathcal{E} = ((u^i)_{i\in\mathcal{I}}, (x^i)_{i\in\mathcal{I}})$. This is precisely the Second Welfare Theorem.

To obtain the First Welfare Theorem, we go the other way. Observe that if endowments e and prices p are given and each agent maximizes utility, it must be the case at the solution consumption bundles $x^1, ..., x^I$, (2) holds and each consumer's budget constraint is satisfied. Then consider the Pareto problem with $\overline{u}^i = u^i(x^i)$ for agents $2, ..., I$. It is easy to check that (1) and each of the constraints is satisfied at $x^1, ..., x^I$ if we define $\mu_l = p_l$, $\lambda^i = 1/\nu^i$, and $\overline{u}^i = u^i(x^i)$. Therefore any Walrasian equilibrium is Pareto optimal.

Finally, there is an alternative approach to characterizing Pareto efficient allocations that is sometimes useful. In this approach, one considers maximizing a linear (Bergson-Samuelson) social welfare function of the form $\sum_i \beta^i u^i$ subject to a resource constraint. The program is:

$$\max_{x^1,...,x^I} \sum_i \beta^i u^i(x_1^i, ..., x_L^i)$$
$$\text{s.t.} \ \sum_i x^i \leq \sum_i e^i$$

Given monotonicity of utility functions, the resource constraint will bind at the optimum and the additional Kuhn-Tucker condition for optimality is that for all agents i and commodities l:

$$\beta^i \frac{\partial u^i}{\partial x_l^i} - \delta_l \leq 0 \qquad x_l^i \geq 0 \qquad \left(\beta^i \frac{\partial u^i}{\partial x_l^i} - \delta_l\right) \cdot x_l^i = 0 \tag{3}$$

Letting $\beta^i = \lambda^i$ and $\delta_l = \mu_l$ we have an exact correspondence between (1) and (3). Letting $\beta^i = 1/\nu^i$ and $\delta_l = p_l$, we have an exact correspondence between (2) and (3). So not only do Pareto optimal allocations coincide with Walrasian equilibrium allocations coincide in the sense of the welfare theorems, they coincide with allocations that maximize a linear social welfare function.

6 Existence of Equilibrium

For nearly a hundred years after Walras wrote down his model of general equilibrium, it was an open question as to whether such an equilibrium actually existed. Early approaches to proving existence results focused on a general equilbrium model due to Cassel (1924), which took as its basic premises aggregate demand for each commodity as a function of all commodity prices (so no individual utility maximization), and a simple supply side model where each commodity could be produced from a fixed resource input and each firm would produce at zero profits (so very simple linear production functions). Equilibrium was defined as a set of commodity prices and quantities such that demand just equaled supply for each commodity.

Ignoring the supply side for a moment, and letting $x_i(p_1, ..., p_n)$ denote the aggregate demand for good i as a function of prices, the basic problem was to show the existence of a price vector $p_1, ..., p_n$ satisfying:

$$x_i(p_1, ..., p_n) = e_i \text{ for all } i = 1, ..., n.$$

The basic idea in the early literature was to count up equations and unknowns. Unfortunately, this led to some confusion about what would happen if the solution to the equations involved either negative prices or quantities.

In 1951, John Nash published his Princeton dissertation in which he used a fixed point theorem to prove the existence of Nash equilibrium in games. Once this idea was out, general equilibrium theorists realized how to provide general existence proofs for Walrasian equilibrium. The big breakthrough came when Arrow and Debreu (1954) teamed up to prove the following result.

Theorem 3 (Existence of Walrasian Equilibrium) *(Arrow-Debreu, 1954) Given an economy $\mathcal{E}$ satisfying (A1)–(A4), there exists a Walrasian equilibrium (p, x).*

The proof is pretty involved and arguably not all that enlightening, but this has been such a persistent question in modern economics that we'd be remiss not to attempt it. The general fixed point style of proof is also common in other problems. What we'll do here is start with a fairly simple and intuitive proof for the case of two commodities, then give a more general proof using a fixed point theorem.

6.1 Excess Demand Functions

As a starting point, we're going to introduce the idea of an *excess demand* function.

Definition 4 *The* ***excess demand function of agent*** i *is:*

$$z^i(p) = x^i(p, p \cdot e^i) - e^i$$

where $x^i(p, p \cdot e^i)$ *is* i*'s Walrasian demand function. The* ***aggregate excess demand function*** *is:*

$$z(p) = \sum_i z^i(p).$$

From the definition of the excess demand function, it should be clear that if a price vector $p \in \mathbb{R}^L_+$ satisfies $z(p) = 0$, then $(p, (x^i)_{i \in \mathcal{I}})$ will be a Walrasian equilibrium if x^i is defined to be i's Marshallian demand given the price vector p. Why? Because $(p, (x^i)_{i \in \mathcal{I}})$ will then satisfy both the individual optimization part of the definition of equilibrium (by definition of x^i) and market clearing (by the fact that $z(p) = 0$).

Thus, proving existence of equilibrium boils down to establishing that a solution to $z(p) = 0$ exists given our assumptions (1)–(4). From the first part of this class on consumer theory, we have the following properties of the excess demand function.

Proposition 4 *Suppose (A1)–(A4) are satisfied. Then the aggregate excess demand function* $z(p)$ *satisfies:*

(i) z *is continuous;*

(ii) z *is homogenous of degree zero;*

(iii) $z(p) = 0$ *for all* p *(Walras' Law);*

(iv) for some $Z > 0$, $z_l(p) > -Z$ for every $l \in \mathcal{L}$ and all p.

(v) if $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then $\max\{z_1(p^n), ..., z_L(p^n)\} \to \infty$.

Proof. Except for the last property, these all follow directly from properties of the Marshallian demand function established in the first half of the class. The last property isn't complicated. As some, but not all, prices go to zero, there must be some consumer whose wealth is not going to zero. Because he has strongly monotone preferences, he must demand more and more of one of the goods whose price is going to zero. *Q.E.D.*

In the next section, we use these properties to establish existence of equilibrium for the two good case. The proof won't be entirely general because we're going to treat $z(\cdot)$ as a function. In general, recall that agents might not have a unique optimal bundle given a set of prices, so Marshallian demand, and hence $z(\cdot)$ should really be treated as a correspondence. We'll deal with that in the following section.

6.2 An Intuitive Argument

To gain some intuition, let's consider the case where there are only two goods in the economy, so we want to find a Walrasian equilibrium price vector $p = (p_1, p_2)$ with $z(p) = 0$. Because $z(\cdot)$ is homogenous of degree zero, we can safely normalize the price of $p_2 = 1$, meaning we can search only over price vectors $p = (p_1, p_2 = 1)$. Moreover, because of Walras' Law, $z(p) \cdot p = 0$ for *any* p, so to establish an equilibrium, it suffices to find a price p_1 such that $z_1(p_1, 1) = 0$. (If this holds, then $z_2(p_1, 1) = 0$ by Walras' Law.)

Figure 6 graphs $z_1(p_1, 1)$ as a function of p_1. There are three important points to note in the picture that must hold. First, $z_1(\cdot, 1)$ is continuous as we showed above. Second, for very small values of p_1, $z_1(p_1, 1)$ is strictly positive. Third, for very large values of p_1, $z_1(p_1, 1)$ is negative. There must be at least one value of p_1 for which $z_1(p_1, 1) = 0$ and the vector $(p_1, 1)$ is a Walrasian equilibrium price vector.

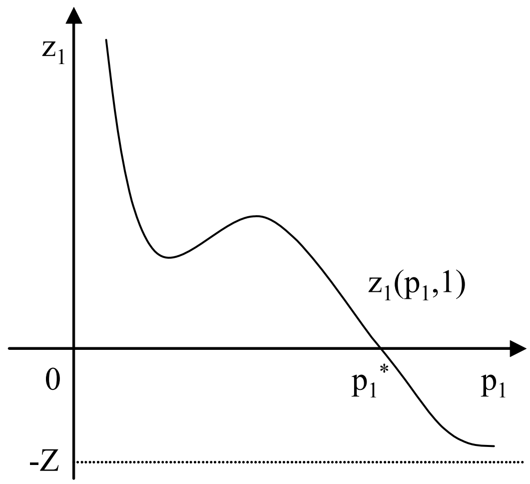


Figure 6: Existence of Walrasian Equilibrium with Two Goods

The only subtlety to this argument is establishing points 2 and 3, namely that when one of the goods becomes infinitely cheap relative to the other good, there will be excess demand for the cheaper good. Formally, this follows from conditions *(iv)* and *(v)* in the above Proposition. For very small values of p_1, condition *(v)* implies that either z_1 or z_2 must be very large. It can't be, however, that excess demand for the relatively expensive good 2 is very large however, because then $p_2 \cdot z_2 = z_2$ would be very large, and condition *(iv)* implies that $p_1 \cdot z_1 > -Z$ for some fixed Z. Hence Walras' law stating that $p_1 \cdot z_1 + p_2 \cdot z_2 = 0$ would be violated. A symmetric argument implies that for very large values of p_1, $z_1(p_1, 1)$ must be negative.

6.3 Proof of Equilibrium Existence

At this point, we're ready to take a shot at proving the existence theorem. As suggested above, you'll need some familiarity with correspondences (consult MWG or the Useful Math notes if in doubt).

The proof is going to work a bit differently from the two-good case in the following sense. Rather than look for a price vector p that solves $z(p) = 0$ (with the allocation being left implicit in the definition of $z(\cdot)$) we're going to define a map Ψ that takes the set of price-aggregate demand pairs (p, x) into itself. The map Ψ will be defined so that any fixed point of Ψ will be a Walrasian equilibrium. Then we'll establish that a fixed point exists.

The map Ψ is going to be defined as follows. Given a price-aggregate demand pair (p, x), we define the new aggregate demand by letting agents optimize given prices. That is, agents take prices as given and have a Marshallian demand correspondence (function in the case of strict concavity but correspondence if indifference curves are flat and many points on the budget line give the same utility). We establish that aggregate demand is a non-empty, convex valued and upper semi-continuous (usc) correspondences of prices.

To get a new set of prices, we apply a trick that was first introduced by Debreu. We have an additional agent called the 'price-player'. He takes the old aggregate demand as given and sets prices in a clever way; his choice correspondence is an usc, convex-valued and non-empty correspondence of aggregate demand. An important property will be that if at the old prices, the agent's demands just clear the market, it will be optimal for the price player not to change prices.

With this set-up we put together all the choices into a correspondence Ψ that maps a price vector and an aggregate demand into new prices (chosen by the price player) and a new aggregate demand (chosen by the agents). As noted, Ψ is a correspondence, not a function. We then apply Kakutani's Theorem, which asserts that under conditions we've established for Ψ, it must have a fixed point (for this, we'll need to state and maybe understand Kakutani's theorem). We then argue that a fixed point of our map corresponds exactly to a Walrasian equilibrium. The whole thing sounds complicated, but hopefully it won't be too bad.

6.4 Two Math Theorems

We're going to need to appeal to two mathematical theorems during the proof. The first is Kakutani's fixed point theorem (it's hard to prove).

Theorem 5 (Kakutani) *Suppose $A \subset \mathbb{R}^n$ is a convex, closed and bounded set. Suppose $f : A \Rightarrow A$ is a correspondence which is convex valued, non-empty valued* *for all $x \in A$ and which is upper-hemi-continuous (uhc). Then there exists a $x \in A$ such that $x \in f(x)$.*

If you know Brouwer's fixed point theorem for functions, this result is quite similar. If you don't know Brouwer's fixed point theorem, I'll try to give some intuition for it in class. If this theorem just looks totally mysterious, just take it as given and don't worry too much about it.

The second theorem (not hard to prove) is called 'maximum principle' or sometimes Berge's theorem (see M-W-G Theorem M.K.6)

Proposition 6 *Suppose we have a continuous correspondence $C : Q \Rightarrow \mathbb{R}^N$, with $c(q)$ compact and non-empty for all $q \in Q$ and a continuous function $f : \mathbb{R}^N \to \mathbb{R}$. Consider a maximization problem $\max f(x) s.t. x \in c(q)$. The maximizer correspondence will be upper-hemi-continuous. The value function will be continuous.*

6.5 Proof of the Existence Theorem

We'll prove the theorem is a series of steps.

STEP 1 (Normalize Prices):

Recall that what matters in the Walrasian model is relative prices, so we are always free to normalize one of the prices. Rather than set $p_1 = 1$, however, it's convenient to normalize the prices so that they all sum to 1. Define:

$$\Delta = \left\{ p \in \mathbb{R}^L_+ : p_1 + ... + p_L = 1 \right\}$$

to be the set of price vectors that sum to one (the price simplex).

STEP 2 (Aggregate Demand):

We first define individual Marshallian demands in such a way that they are usc in prices. In order to agents' demand correspondences are usc using Berge's Theorem, we face the slight problem that the budget correspondence $\mathcal{B}^i(p)$ isn't compact valued at prices on the boundary of Δ. Therefore we cleverly define the compact set

$$T = \{x \in \mathbb{R}^L_+ : x \leq 2 \sum_{i \in \mathcal{I}} e^i\}$$

and consider for each agent $i \in \mathcal{I}$ the correspondence

$$\psi^i(p) = \arg \max_{c \in \mathcal{B}^i(p) \cap T} u^i(c)$$

The correspondence ψ^i is non-empty valued and usc for each agent i. Moreover, because $u^i(.)$ is concave, ψ^i is convex-valued. Note that our assumption $e^i \gg 0$ is crucial here. If $e^i_l = 0$ for some $l \in \mathcal{L}$ the budget correspondence will not be continuous at $p_l = 0$ (which one fails, usc or lsc ?), so we can't apply Berge's Theorem.

Finally, we define the aggregate demand correspondence:

$$\Psi^D(p) = \sum_{i \in \mathcal{I}} \psi^i(p) = \left\{ x : \exists x^1 \in \psi^1(p), ..., x^I \in \psi^I(p) \text{ s.t. } x = \sum_{i \in \mathcal{I}} x^i \right\}.$$

It's easy to check that $\Psi : \Delta \to T$ is a non-empty, convex-valued and usc in prices, given what we know about $\psi^1, ..., \psi^I$.

STEP 3 (The Price Player):

Now we introduce the price player, whose correspondence is defined as $\Psi^P : T \Rightarrow \Delta$, where

$$\Psi^P = \arg \max_{p \in \Delta^{L-1}} p \cdot (x - e)$$

where $e = \sum_{i \in \mathcal{I}} e^i$ is the aggregate endowment. That is, the price player chooses new prices to maximize the value of the aggregate excess demand (at the old prices). Note that Ψ^P is non-empty, convex-valued and usc.

STEP 4 (The Fixed Point):

Define $\Psi : \Delta \times T \Rightarrow \Delta \times T$ by:

$$\Psi(p, x) = (\Psi^P(x), \Psi^D(p)).$$

Because the product of non-empty and convex-valued usc correspondeces is itself non-empty, convex-valued and usc, we can apply Kakutani's theorem. This establishes a fixed point $(p^*, x^*) \in \Psi(p^*, x^*)$.

STEP 5 (The Walrasian Equilibrium):

We now argue that from (p^*, x^*) we have a W.E., or more precisely that p^* is a Walrasian equilibrium when paired with the individual demands $x^{1*}, ..., x^{I*}$ that make up x^*.

In particular, because $x^* \in \Psi^D(p)$, there exist $x^{1*}, ..., x^{I*}$ summing to x^* with the property that $x^{i*} \in \arg\max_{c \in \mathcal{B}^i(p^*) \cap T} u^i(c)$. For the individual optimization part of equilibrium, we need to verify that $x^{i*} \in \arg\max_{c \in \mathcal{B}^i(p^*)} u^i(c)$. For this, note that $p^* \in \Psi^P(x^*)$:

$$0 \geq p^* \cdot (x^* - e) \geq p \cdot (x^* - e) \qquad \text{for all } p \in \Delta.$$

The latter inequality implies that $x^* - e \leq 0$ and in particular $x^{i*} < 2e$. Therefore $x^{i*} \in \arg\max_{c \in \mathcal{B}^i(\bar{p})} u^i(c)$ because if there were a $c \in \mathcal{B}^i(p)$ with $u^i(c) > u^i(x^{i*})$ then for small $\lambda > 0$, $\lambda c + (1 - \lambda)x^{i*} \in \mathcal{B}^i(p) \cap T$ and by concavity of u^i, $u^i(\lambda c + (1 - \lambda)x^{i*}) > u^i(x^{i*})$, a contradiction.

It remains to show market clearing: i.e. that $x^* = e$. By Walras Law, we have $p^* \cdot x^* = p^* \cdot e$. Therefore if $x_l^* - e_l < 0$ for some good l, we must have $p_l^* = 0$ by the price player's optimization. But then we can simply replace x_l^{*1} by $x_l^{*1} - (x_l^* - e_l)$ and get market clearing. *Q.E.D.*

7 Uniqueness, Stability and Comparative Statics

We now turn to a brief discussion of three important questions in general equilibrium theory. These are:

1. Is there a unique Walrasian equilibrium? If not, how many Walrasian equilibria are there?

2. Is the Walrasian equilibrium stable in the sense that reasonable dynamic adjustment processes converge to equilibrium prices and allocations?

3. Does Walrasian equilibrium impose meaningful restrictions on observable data? For instance, what can we say about how a change in endowments will change equilibrium prices?

As we've already suggested, the first two questions have essentially negative answers. Generally speaking there can be a lot of Walrasian equilibria for a given specification of preferences and endowments (though not an infinite number). There is also no particular reason, without strong assumptions on prefernces, to believe that dynamic adjustment processes will converge to a Walrasian equilibrium outcome. In contrast, the third question will have a positive answer. If we observe data on endowments and prices for a fixed set of agents trading at equilibrium prices, and then are asked to predict equilibrium prices and quantities for these same agents after a change in endowments, we will generally be able to say something (though maybe not all that much) about what the new equilibrium prices and quantities will be.

7.1 Global Uniqueness

The Edgeworth box picture we drew above in Figure 4 suggests strongly that there need not be a unique Walrasian equilibrium. The following simple numerical example (from MWG) shows that the picture is not at all pathological.

Suppose there are two goods and two consumers with utility functions:

$$u^1(x_1^1, x_2^1) = x_1^1 - \frac{1}{8}(x_2^1)^{-8} \qquad\qquad u^2(x_1^2, x_2^2) = -\frac{1}{8}(x_1^2)^{-8} + x_2^2.$$

Both utility functions are quasi-linear, but with respect to different numeraires. Assume the endowments are $e^1 = (2, r)$ and $e^2 = (r, 2)$, where $r = 2^{8/9} - 2^{1/9}$ (this is just to make everything work out nice). The Marshallian demands at prices p_1, p_2 are:

$$\begin{aligned} x^1(p_1, p_2) &= \left(2 + r\left(\frac{p_2}{p_1}\right) - \left(\frac{p_2}{p_1}\right)^{8/9}, \left(\frac{p_2}{p_1}\right)^{-1/9}\right) \\ x^2(p_1, p_2) &= \left(\left(\frac{p_1}{p_2}\right)^{-1/9}, 2 + r\left(\frac{p_1}{p_2}\right) - \left(\frac{p_1}{p_2}\right)^{8/9}\right). \end{aligned}$$

Normalizing the price of good 2 so that $p_2 = 1$, we can write the aggregate excess demand curve for good 1 as

$$z_1(p_1, 1) = x_1^1(p_1, 1) + x_1^2(p_1, 1) - (2 + r)$$

or

$$z_1(p_1, 1) = r\left(\frac{1}{p} - 1\right) - \left(\frac{1}{p}\right)^{8/9} + \left(\frac{1}{p}\right)^{1/9}$$

This is pictured in Figure 7 below. Note that the excess demand function has three zeros: at $p_1 = 1/2$, at $p_1 = 1$ and at $p_1 = 2$. All three correspond to Walrasian equilibrium. Another way to see the multiplicity is to look at Figure 4 above, where this model is depicted in the Edgeworth box.

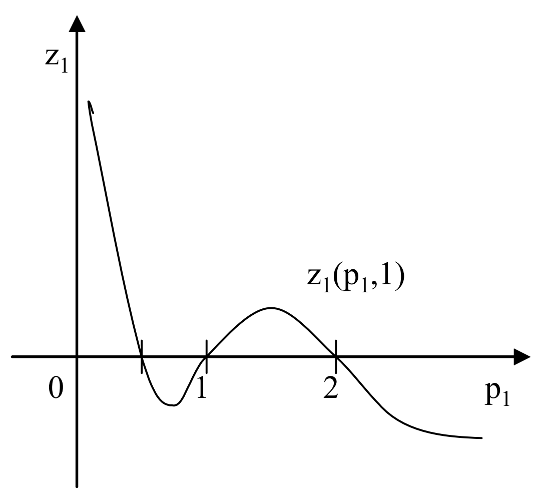


Figure 7: Multiple Walrasian Equilibria

What should be clear from both figures is that multiple equilibria do not always arise: for a given set of endowments and preferences, marshallian demands have to change in the right way with prices to get multiplicity. On the other hand, neither are multiple equilibria all that special — in the numerical example, we picked the numbers to come out nicely, but a small change in preferences or endowments won't upset the fact that there are three equilibria.

This being said, a good deal of effort has gone into identifying conditions on preferences that rule out multiplicity and ensure uniquess. We will discuss one such condition, the gross substitutes property, below.

7.2 Local uniqueness

Even if there are multiple Walrasian equilibria for a given set of preferences and endowments, it may still be the case that each of these equilibria are all *locally unique* in the sense that there is no other Walrasian equilibrium price vector within a small enough range around the orginal equilibrium price vector.

It turns out that for "most" economies, the Walrasian equilibria are locally unique. As a consequence the set of Walrasian equilibria is finite. To understand this idea, it's again useful to think about the two commodity case, where we can normalize $p_2 = 1$ and look for values of p_1 that satisfy $z_1(p_1, 1) = 0$. In the example above, shown in Figure 7, there are three Walrasian equilibria. Each of the equilibria, however, is locally unique. An equilibrium is not locally unique if its price vector p is the limit of a sequence of other equilibrium prices. An example of local non-uniqueness would be if $z_1(p_1, 1)$ was equal to zero over some interval of prices $[p_1^*, p_1^{**}]$. This can happen, as is shown in Figure 8. The point to realize from this figure, however, is that such an occurrence is extremely special. Any small perturbation of $z_1(\cdot, 1)$, such as would arise from a small change in the endowments, will restore us to the case of having a finite number of locally unique equilibria.[2]

[2] Note that the picture also suggests something more, which is that because $z_1(\cdot, 1)$ starts above zero and finishes below zero, "generically" there should be an odd number of equilibria. This turns out to be correct and can be shown in the many commodity case as well.

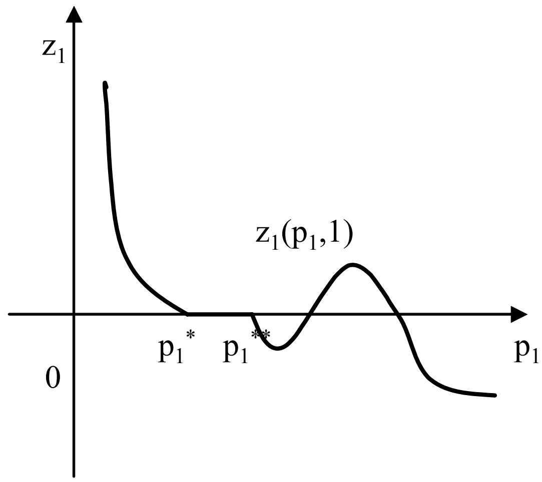


Figure 8: Local Non-uniqueness of equilibrium

This is roughly speaking the kind of argument one uses in the more general many-commodity case to prove that the equilibria of "typical" economies are locally unique. Heavier math comes into play, however, so we'll skip the formal analysis. This is often a topic covered in advanced general equilibrium classes.

7.3 Tatonnement Stability

So far we haven't touched the question of where prices come from and whether it is reasonable to expect to see Walrasian prices in a given economy – even if such prices exist and are unique. It turns out that the theory of general equilibrium is quite weak on the kinds of price formation processes that might lead to Walrasian outcomes.

Walras himself suggested the following kind of price adjustment process that he called "tatonnement" (french for "groping"). Imagine that the agents meet in a public square and there is a "Walrasian auctioneer" who calls out prices. After he does this, everyone calls out their demands at those prices. The auctioneer then adjusts prices and calls out a new set. The process continues until a set of prices is called out for which demand just equals supply. At this point the auctioneer stops, announces prices, and trade occurs.

A candidate price adjustment rule for the auctioneer is:

$$p(t+1) = p(t) + \alpha z(p(t)) \qquad \text{for small } \alpha > 0.$$

Clearly the only stationary points of this process are prices p at which $z(p) = 0$, i.e. Walrasian equilibrium prices. Moreover, an equilibrium price vector p could naturally be said to be *locally stable* if the price adjustment rule converges to p from any "nearby" starting prices. If an equilibrium price vector p is stable a small perturbation away from p will not have a long-run effect if the auctioneer begins again to look for an equilibrium. Finally, an equilibrium price vector p is (globally) *stable* if the price adjustment rule converges to p from any initial prices.

Walrasian tatonnement gives us a way to study how equilibrium prices might be reached, but the model has obvious drawbacks. First, no trade actually takes place at the non-equilibrium prices. Second, even if one were able to organize this giant procedure, people might not want to announce their true demands. Finally a third drawback, and the one that delivered a huge blow to this line of research is that Walrasian tatonnement can cycle without converging to an equilibrium. This was first shown in a famous paper by Scarf (1960), who provided examples where both local and global stability failed. Prior to this paper, all that had been shown were assumptions on preferences that did ensure stability of Walrasian equilibrium prices (e.g. Arrow and Hurwicz, 1958).

7.4 Debreu-Sonnenschein-Mantel Theorem

Above we posed the questions of whether Walrasian equilibrium is unique and whether it is stable under reasonable dynamic adjustment processes. The answer to both turned on the structure of the aggregate excess demand function of the economy: $z(\cdot)$. Walrasian equilibrium is unique if there is a unique solution to $z(p) = 0$, and stable if the zero is a stable point of $z(\cdot)$.

This leads us to ask what exactly we know about the structure of aggregate excess demand. Above, we proved that $z(\cdot)$ is continuous, homogenous of degree zero in prices, satisfies Walras' Law, and has certain boundary properties: in particular as $p \to 0$, $z(p) \to \infty$.

In a famous paper, Hugo Sonnenschein (1973) asked whether there are any further restrictions on $z(\cdot)$ that can be derived from the assumption of consumer maximization. Remarkably, Sonnenschein (1973), Debreu (1974) and Mantel (1974) were able to show that the answer is "no". This gives something of a negative conclusion to our original questions as it implies that given an economy that has an equilibrium at a certain price vector p, it is possible for that economy to have an arbitrary number of equilibria with arbitrary stability properties in an arbitrarily small neighborhood of prices around p.

Theorem 7 *(Sonnenschein-Mantel-Debreu) Suppose we have an open and bounded subset $B \subset \mathbb{R}^L_{++}$ and a continuous function $f(p) : B \to \mathbb{R}^L$ satisfying homogeneity of degree zero and Walras' Law. Then there exists an economy $\mathcal{E}$ with aggregate excess demand function $z(p)$ satisfying $f(p) = z(p)$ on B.*

Proof. We'll skip it. You can look up the $L = 2$ case in MWG. *Q.E.D.*

A common interpretation of this theorem (as in MWG) is that "anything goes" in general equilibrium theory. That is, that without very special assumptions (like Cobb-Douglas preferences or something like that): (i) pretty much any comparative statics result could be obtained in a general equilibrium model, and (ii) general equilibrium theory has essentially no empirical content. We'll see in the next section that this is not quite right.

7.5 Brown-Matzkin Theorem

The Sonnenschein-Debreu-Mantel Theorem says that the aggregate excess demand function has only minimal properties. An implication is that utility maximization imposes no testable restrictions on equilibrium prices. This suggests that one could not test the hypothesis that agents were or were not trading in a Walrasian fashion by observing price data, unless one also made some assumptions about the preferences of the agents who were trading.

A striking result due to Brown and Matzkin (1996), however, says that if one is able to observe endowments as well as prices, then the Walrasian model is testable. That is, there are endowment and price pairs $(p, (e^i)_{i \in \mathcal{I}})$ and $(\hat{p}, (\hat{e}^i)_{i \in \mathcal{I}})$ such that if p is a set of Walrasian prices given a fixed set of agents with endowments $(e^i)_{i \in \mathcal{I}}$, then if this same set of agents has endowments $(\hat{e}^i)_{i \in \mathcal{I}}$, $\hat{p}$ could not possibly be a Walrasian equilibrium price vector. The argument relies on revealed preference.

Theorem 8 *(Brown-Matzkin, 1996) There exist prices and endowments* $(p, (e^i)_{i \in \mathcal{I}})$ *and* $(\hat{p}, (\hat{e}^i)_{i \in \mathcal{I}})$ *such that it is impossible to find monotone preferences* $(u^i)_{i \in \mathcal{I}}$ *with the property that* p *is a Walrasian equilibrium price vector for the economy* $(u^i, e^i)_{i \in \mathcal{I}}$ *and* $\hat{p}$ *is a Walrasian equilibrium price vector for the economy* $(u^i, \hat{e}^i)_{i \in \mathcal{I}}$.

Proof. We use an Edgeworth box example to prove the Theorem for the case of two consumers and two goods. Consider the two Edgeworth boxes in Figure 9. Because p is an equilibrium given e, agent 1 must weakly prefer some bundle on the line segment A to any bundle on the line segment B. By monotonicity, for every point on the line segment $\hat{A}$, there is some point on B that agent 1 strictly prefers. So there is some bundle on A that is preferred to every bundle on $\hat{A}$. Now, if $\hat{p}$ is an equilibrium given $\hat{e}$, we have an immediate contradiction because every bundle on A is available, yet the agent chooses a bundle on $\hat{A}$. *Q.E.D.*

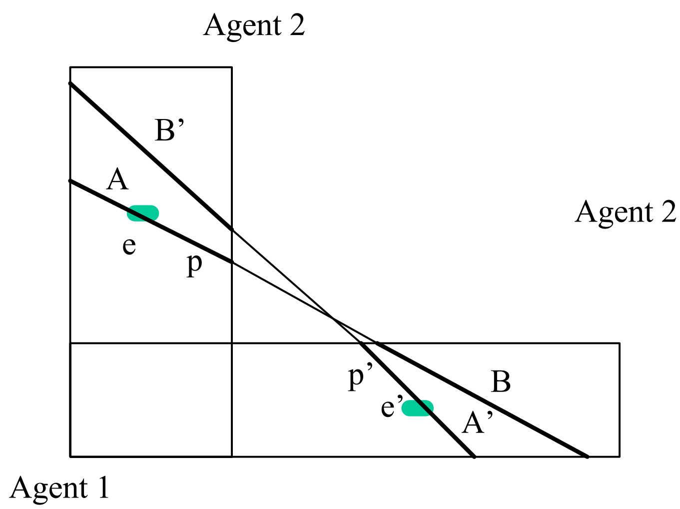


Figure 9: Testable Restrictions of Equilibrium

8 Gross Substitutes

In this section, we consider one particular class of economies — those satisfying the *gross substitutes property* — in which it is possible to get affirmative answers to the uniquencess and stability questions posed above. We then show that economies satisfying gross substitutes also have very nice comparative statics properties.

Two commodities are said to be gross substitutes if an increase in the price of good k increases the demand for good l. More generally, a demand function satisfies the gross substitutes property if an increase in the price of good k increases the demand for every other good l.

Definition 5 *A Marshallian demand function $x(p)$ satisfies the* ***gross substitutes*** *property if, whenever p and p' are such that $p'_k > p_k$ and $p'_l = p_l$ for all $l \neq k$, then $x_l(p') > x_l(p)$ for all $l \neq k$.*

Note that I have stated the condition as requiring a strict increase in the demand for each good l. One can also work with the *weak gross substitutes* property which requires only a non-decrease in the demand for each good l. To keep things simple, we'll stick with the strict case.

We now have the following observation.

Remark 1 *If each individual has a Marshallian demand function satisfying gross substitutes, then both the individual and aggregate excess demand functions satisfy gross substitutes.*

If aggregate demand satisfies the gross substitutes property, then there is a unique Walrasian equilibrium.

Proposition 9 *If the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes, the economy has at most one Walrasian equilibrium, i.e. $z(p) = 0$ has at most one (normalized) solution.*

Proof. Suppose by way of contradiction that $z(p) = z(p') = 0$ for two price vectors p and p' that are not collinear. By homogeneity of degree zero, we can always normalize the price vectors in such a way that $p'_l \geq p_l$ for all $l \in \mathcal{L}$ and $p'_k = p_k$ for some good k. Then to move from p to p', we can think about moving in a series of $n-1$ steps, increasing the prices of each of goods $l \neq k$ in turn. At each step where a component of price increases strictly (and there must be at least one such step), the aggregate demand for good k must strictly increase, so that $z_k(p') > z_k(p) = 0$, yielding a contradiction. *Q.E.D.*

It is also possible to show that under the gross substitutes property, Walrasian tatonnement will converge to the unique equilibrium. One approach to showing this is via the following Lemma.

Lemma 10 *Suppose that the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes and that $z(p^*) = 0$. Then for any p not collinear with p^*, $p^* \cdot z(p) > 0$.*

Proof. Let's just consider the proof for the case of two commodities; there's probably an elegant short proof for the general case, but the only proof I could come up with is rather long. With two commodities, let's normalize the price of good 2, so that $p_2^* = p_2 = 1$. Then:

$$\begin{aligned} p^* \cdot z(p) &= (p^* - p) \cdot (z(p) - z(p^*)) \\ &= (p_1^* - p_1) \cdot (z_1(p) - z_1(p^*)) > 0 \end{aligned}$$

The first equality uses Walras' Law $p \cdot z(p) = 0$ and the fact that p^* is an equilibrium so $z(p^*) = 0$. The second uses the price normalization. The final inequality follows because by the gross substitute property, $p_1 > p_1^*$ implies $z_1(p) < z_1(p^*)$ and similarly $p_1 < p_1^*$ implies $z_1(p) > z_1(p^*)$. *Q.E.D.*

This Lemma is similar to the weak axiom of revealed preference that we alluded to in the first half of the class. Marshallian demand is said to satisfy the weak axiom of revealed preference if for any two price vectors p and p':

$$(p - p') \cdot (x(p) - x(p')) \leq 0. \tag{4}$$

This is a pretty strong condition and isn't implied by the gross substitutes property (nor does it imply gross substitutes). Gross substitutes, however, does imply a version of WARP. It implies that the weak axiom holds if one compares p^*, the unique equilibrium price vector, to any other price vector p.

With this Lemma, we can prove the following result about stability.

Proposition 11 *Suppose that the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes and that p is a Walrasian equilibrium price vector. Then the tatonnement adjustment process $dp/dy = \alpha z(p(t))$, with $\alpha > 0$, converges to the relative prices of p as $t \to \infty$ for any initial condition $p(0)$.*

Proof. To prove the result, we show that the "distance" between $p(t)$ and p^* decreases monotonically as time progresses. Let $D(p) = \frac{1}{2}\sum_l (p_l - p_l^*)^2$ denote the distance between p and p^*. Then:

$$\begin{aligned} \frac{dD(p(t))}{dt} &= \sum_l (p_l(t) - p_l^*) \frac{dp_l(t)}{dt} \\ &= \alpha \sum_l (p_l(t) - p_l^*) z_l(p(t)) \\ &= -\alpha p^* \cdot z(p) \leq 0. \end{aligned}$$

Note the use of Walras Law in deriving the third equality. The last inequality is strict unless p is proportional to p^*. Now, because $D(p(t))$ is decreasing monotonically over time, it must converge, either to zero or to some positive number. In the former case, $p(t) \to p^*$. In the latter case, $p(t) \nrightarrow p^*$ but $dD(p(t))/dt \to 0$. The only way this can happen is if $p(t)$ becomes nearly proportional to p^* as $t \to \infty$. But this means that the relative prices of $p(t)$ converge to those of p^* as $t \to \infty$. *Q.E.D.*

Finally, economies with gross substitutes have nice comparative statics properties. In particular, any change that raises the excess demand for good k will increase the equilibrium price of that good. As an example, suppose there are two goods, and normalize $p_2 = 1$. Suppose also that good 1 is a normal good for all agents. Now consider an increase in the endowment of the numeraire good 2 for some of the agents. For any price p_1, this change will increase aggregate demand for good 1 and hence increase aggregate excess demand. As shown in Figure 10, this will shift up the excess demand curve $z_1(\cdot, 1)$ — in the figure the original excess demand curve is denoted by $z_1(\cdot, 1; L)$; the new excess demand curve is $z_1(\cdot, 1; H)$. Because $z_1(\cdot, 1; L)$ is continuous and crosses zero only once (remember equilibrium is unique), the new equilibrium must have a higher price for good 1.

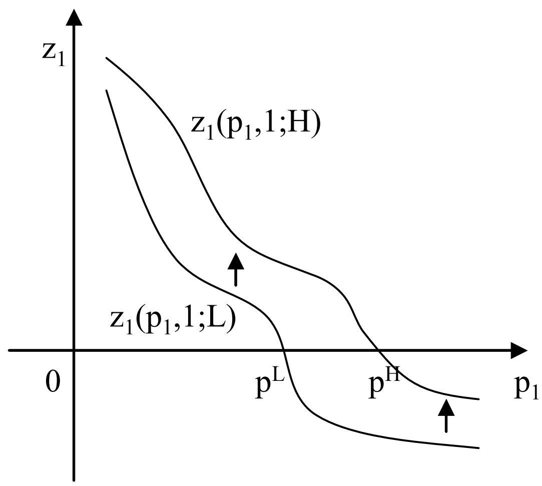


Figure 10: Comparative Statics with Gross Substitutes

This argument can be generalized to the many commodity case — you may see something like this on a problem set.

9 Production in General Equilibrium

Everything we have done so far has been for the special case of an exchange economy where goods simply come from nowhere as endowments. Fortunately, it's pretty easy to incorporate firms and production into our general equilibrium model, so long as we assume: (1) no increasing returns to scale; and (2) perfectly competitive price-taking firms.

In this section, we outline the more general Arrow-Debreu model with production, revisit the welfare theorems and equilibrium existence, and then consider some simple examples.

9.1 Adding Production to the Model

We retain the consumers $i = 1, ..., I$ of the earlier model, with utility functions $u^1, ..., u^I$. We add K firms $k \in \mathcal{K}$ with production sets $Y^k \in \mathbb{R}^N$. Each Y^k is a set of production plans: if $y \in Y^k$, then $y_l < 0$ means good l is being used as an input; $y_l > 0$ means good l is being produced as an output. The firms are owned by the households. We let α^{ki} denote i's share of firm k. A production economy is then:

$$\mathcal{E} = \left((u^i, e^i, (\alpha^{ki})_{k\in\mathcal{K}})_{i\in\mathcal{I}}, (Y^k)_{k\in\mathcal{K}}\right).$$

Firm k takes prices $p \in \mathbb{R}^N$ as given and choose a production plan $y^k \in Y^k$ to solve:

$$\max_{y\in Y^k} p \cdot y.$$

Definition 6 *A* ***Walrasian equilibrium*** *is a vector* $(p, (x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}})$ *such that*

1. *Firms maximize profits: for all* $k \in \mathcal{K}$*,*

$$y^k \in \arg\max_{y\in Y^k} p \cdot y$$

2. *Consumers maximize utility: for all* $i \in \mathcal{I}$*,*

$$\begin{aligned} x^i \in \quad & \arg\max_x u^i(x) \\ & s.t.\ p \cdot (x - e^i) - p \cdot \sum_{k\in\mathcal{K}} \alpha^{ki} y^k \leq 0 \end{aligned}$$

3. *Markets clear:*

$$\sum_{i\in\mathcal{I}} (x^i - e^i) - \sum_{k\in\mathcal{K}} y^k = 0.$$

9.2 Assumptions about Production

We'll want to make some assumptions on Y^k to ensure that an equilibrium exists with production. The simplest such assumption is that Y^k is convex and compact for all firms k, but it seems unreasonable to assume that a production set is bounded. Instead, we assume:

(A5) For all firms $k \in \mathcal{K}$, Y^k is closed and convex.

(A6) For all firms $k \in \mathcal{K}$, $0 \in Y^k$ and $\mathbb{R}^N_{--} \subset Y^k$.

Note that these assumptions rule out increasing returns to scale. If $y \in Y^k$, then so is βY^k for any $0 < \beta < 1$. So it is always possible to "scale" down production or break it up into arbitrarily small productive units.

We need one further assumption to ensure that firms cannot cooperate in a clever way and produce an infinite amount of goods — i.e. to ensure that one firm doesn't produce 1 pound of iron into 1 pound of steel, while another firm produces 2 pounds of iron from that 1 pound of steel. Debreu (1959) makes an assumption directly on the aggregate production possibilities:

(A7) If $Y = \sum_{k \in \mathcal{K}} Y^k$, then $Y \cap -Y = \{0\}$.

Think about why this rules out the above story. With these assumptions in place, our earlier welfare and existence results carry through.

9.3 Efficiency and Existence

The definition of feasibility and Pareto efficiency carry through immediately to the case of production.

Definition 7 *An allocation and production plan* $((x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}})$ *is **feasible** if* $\sum_{i \in \mathcal{I}}(x^i - e^i) - \sum_{k \in \mathcal{K}} y^k \leq 0$.

Definition 8 *A feasible allocation and production plan* $((x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}})$ *is **Pareto efficient** if there is no other feasible allocation and production plan* $((\hat{x}^i)_{i \in \mathcal{I}}, (\hat{y}^k)_{k \in \mathcal{K}})$ *satisfying* $u^i(\hat{x}^i) \geq u^i(x^i)$ *for all* i*, with strict inequality for at least one* i'*.*

We now state the two welfare theorems.

Theorem 12 (First Welfare Theorem) *Assume* $\mathcal{E}$ *is a production economy that satisfies (A2). If* $(p, (x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}})$ *is a Walrasian equilibrium for* $\mathcal{E}$*, then* $((x^i)_{i \in \mathcal{I}}, (y^k)_{k \in \mathcal{K}})$ *is Pareto efficient.*

The proof is virtually identical to the exchange case. Try to replicate it on your own.

Theorem 13 (Second Welfare Theorem) *Assume utility functions and production sets satisfy (A2)–(A5) and that* $((x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}})$ *is a Pareto efficient allocation. Suppose* $x^i \gg 0$ *for all* $i \in \mathcal{I}$*. Then there is a price vector* $p > 0$*, ownership shares* $(\alpha^{ki})_{i\in\mathcal{I},k\in\mathcal{K}}$*, and endowments* $(e^i)_{i\in\mathcal{I}}$ *such that* $(p, (x^i)_{i\in\mathcal{I}}, (y^k)_{k\in\mathcal{K}})$ *is a Walrasian equilibrium given these endowments and ownership shares.*

The proof again relies on the Separating Hyperplane Theorem; you can check it out in MWG. Note that the key assumption is convexity of the production possibility sets. This is what enables us to find a separating hyperplane between the set of feasible production plans and the aggregate "better than" set. One then shows that the separating hyperplane is a supporting price vector.

What about equilibrium existence? If we impose all three of the Assumptions above, we're in good shape.

Theorem 14 (Existence of Equilibrium) *Assume* $\mathcal{E}$ *is a production economy satisfying (A1)–(A7). Then there exists a Walrasian equilibrium of* $\mathcal{E}$*.*

9.4 Linear Activity Analysis

If we are modeling production, we not only have to pick utility functions but also production sets or production functions. A simple case is the so called 'linear activity model' of production. In this model, all production sets are convex cones spanned by finitely many rays. In particular, there is only one firm (this actually won't make any difference — see below). The firm has access to M linear activities $a_m \in \mathcal{M} \subset \mathbb{R}^L$. It can operate each activity at some level $\gamma \geq 0$. The production set Y is the convex hull of these activities,

$$Y = \{y \in \mathbb{R}^L : y = \sum_{m=1}^{M} \gamma_m a_m \text{ for some } \gamma \in \mathbb{R}^M_+\}.$$

Our assumption of free disposal is satisfied if the vectors

$$(-1, 0, \dots 0), (0, -1, 0, \dots 0), \dots (0, \dots, 0, -1)$$

are all in $\mathcal{M}$.

Figure 11 shows the special case of 4 activities and 2 goods. There are two productive activities: activity 1 allows 2 units of good 2 to be converted into 1 unit of good 1. Activity 2 allows 3 units of good 1 to be converted into 1 unit of good 2. Also there are two "free disposal activities". Therefore:

$$\mathcal{M} = \{(1, -2), (-3, 1), (0, -1), (-1, 0)\}.$$

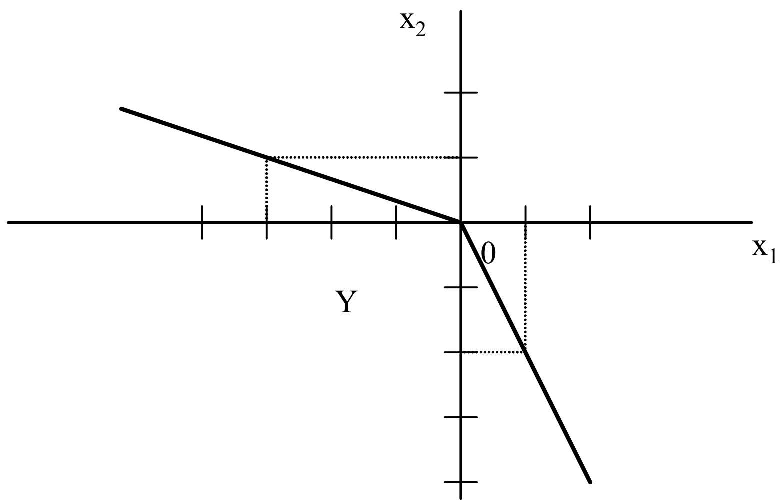


Figure 11: Activity Analysis Model

In the activity analysis model, given a price vector p, a profit maximizing production plan exists if and only if $p \cdot a_m \leq 0$ for all $m = 1, ..., M$. If $p \cdot a_m > 0$ for some $m = 1, ..., M$, the firm could choose $\gamma_m \to \infty$ and make infinite profits. Also, if $p \cdot a_m < 0$ for some m it is clear that the optimal $\gamma_m = 0$.

This simple observation already tells us a lot about what kind of prices could potentially be equilibrium prices. Indeed, in many cases the equilibrium prices will just be determined by the zero-profit conditions, with utility maximization and market clearing pinning down the levels at which the activities are operated.

An important thing to note is that if all production sets are of this simple linear form, firms do not play a role at all. As there will never be any equilibrium profits, what matters is just the aggregate production set. Whether we interpret each activity as a separate firm or we assume that one firm owns all the activities, or even that several firms operate different sets of overlapping activities makes no difference as long as we stay in our competitive paradigm.[3] This constant returns property is shared by the Cobb-Douglas production model that you have probably seen a lot in macroeconomics.

9.5 An Example with Numbers

To make things really concrete, let's consider an example with numbers. Suppose there are two agents and three goods. The agents have identical utility functions:

$$u^i(x) = \log(x_1) + \log(x_2) + \log(x_3)$$

Endowments are $e^1 = (1, 2, 3)$ and $e^2 = (2, 2, 2)$. Suppose that there are two activities $a_1 = (2, -1, 0.5)$ and $a_2 = (0, 1, -1)$.

What does the Walrasian equilibrium look like? Let's normalize $p_3 = 1$. Now, if activity 2 is used in equilibrium, it must be the case that (by zero profit) $p_2 = 1$. Similarly, if activity 1 is used in equilibrium, then $p_1 = 0.25$. These prices are upper bounds on the equilibrium prices if these activities are not used in equilibrium.

Let's see if we can find an equilibrium where both activities are used. Given prices $p = (0.25, 1, 1)$, we solve the utiliy maximization problem for agent i. This gives us:

$$\frac{1}{p_1 x_1^i} = \frac{1}{p_2 x_2^i} = \frac{1}{p_3 x_3^i} \qquad \text{and} \qquad \sum_l p_l x_l^i = \sum_l p_l e_l^i$$

Plugging in our price vector and the endowments, we have:

$$\frac{4}{x_1^i} = \frac{1}{x_2^i} = \frac{1}{x_3^i} \qquad \text{and} \qquad \frac{1}{4} x_1^i + x_2^i + x_3^i = w^i$$

[3] In fact, we can even assume that each agent performs an activity or two himself (household production).

where $w^1 = 5.25$ and $w^2 = 4.5$. Therefore:

$$x^1 = (7, 1.75, 1.75) \qquad\qquad x^2 = (6, 1.5, 1.5)$$

Therefore aggregate demand is $(13, 3.25, 3.25)$. The aggregate endowments are $(3, 4, 5)$, so the only way we can have market clearing is if the aggregate production is $(10, -0.75, -1.75)$. This isn't a problem. The firm will simply operate activity 1 at a level $\gamma_1 = 5$ and operate activity 2 at a level $\gamma_2 = 4.25$.

10 General Equilibrium with Uncertainty

Our goal in this last section is to introduce time and uncertainty into the basic model. Introducing uncertainty allows a role for financial markets. We first discuss the basic framework, then look at a model with financial markets and a single consumption good. In the context of this simple model, we consider what it means for there to be an absence of arbitrage possibilities. We also look at why the first welfare theorem can fail if there are too few financial securities.

10.1 Modeling Uncertainty and Time

Among the many simplifications of the Arrow-Debreu model we have studied so far is that it's essentially a static model with no uncertainty at all. Ideally, we'd like to include both time and uncertainty into our model of competitive trade.

Introducing time into the model isn't too hard. A tomato in summer is a different good than a tomato in winter. So perhaps we can just think about a commodity as being identified not only by its physical characteristics but also by its date.

Uncertainty seems more complicated, but a brilliant modelling innovation of Arrow (1953) comes to the rescue. Arrow's insight was to introduce "states of the world" along the lines of Savage's decision theory. A state of the world is a complete description of a date-event. Unlike in Savage, however, we're going to assume that these states aren't personal and subjective; instead everyone somehow agrees on the possible states (there could be a lot). People don't have to agree on the probabilities of the states occurring, though that is often assumed.

We now think about the general model as having a finite number of time periods. In each period there is a set of possible states and there can be uncertainty about what state will arise at date $t+1$ — the probabilities can even depend on what state was realized at date t.

With these ideas in mind, we can think about re-interpreting our Walrasian model as follows. We model uncertainty as an event tree with S nodes, $\xi \in \Xi$. We done a node's predessor by ξ_{-} and its set of successors by $\Upsilon(\xi)$. At each t we summarize the nodes in this period in a set $\mathcal{N}_t$. We denote the root node by 0. This is pictured in Figure 12.

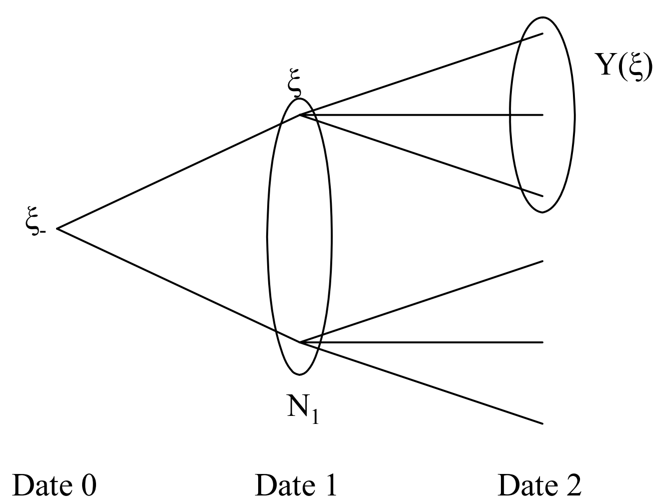


Figure 12: An Event Tree

There are L commodities at every node so the total number of commodities is SL. There are I agents. Each has an endowment $e^i \in \mathbb{R}^{SL}_{++}$. Agent i's consumption set is $\mathbb{R}^{SL}_{+}$ and his utility function is $u^i : \mathbb{R}^{SL}_{+} \to \mathbb{R}$. The utility function may or may not satisfy the von-Neumann Morgenstern axioms.

We define a Walrasian equilibrium exactly as before: a set of prices and resulting allocation such that *(i)* all agents maximize utility given prices; and *(ii)* markets clear. The idea here is that all trades take place at date zero and there is no re-trading in later period. Under our earlier assumptions on preferences, a Walrasian equilibrium exists, and the Welfare Theorems hold.

This is the model in chapter 7 of Debreu's *Theory of Value.* As Debreu puts it: "A contract for the transfer of commodities now specifies, in addition to its physical properties, its location and date, an event on the occurrence of which the transfer is conditional. This new definition of commodity allows one to obtain a theory of uncertainty free from any probability concepts and formally identical to the theory of certainty."

This is quite elegant, but as Arrow originally pointed out, it seems unrealistic that all these contingent trades would occur at date 0. Instead, what tends to happen is that there are financial securities that are traded on exchanges and some of these securities pay out contingent on certain events (e.g. hurricane insurance pays out contingent on there being a hurricane; stocks pay dividends contingent on company performance).

Arrow cleverly reformulated the model as follows. Assume at each node ξ there are spot markets for the L commodities at that node. Assume that these commodities have prices $p(\xi)$. At node 0 there are now $(S-1)$ Arrow securities (i.e. one for each future node), where an Arrow security ξ pays one unit of good one at node ξ. An equilibrium is now defined as utility maximization and market clearing at each node and in the $S-1$ markets for Arrow securities at date 0. The amazing result is that even though there are only $S-1$ securities in the economy, the Arrow-Debreu allocation obtains. The result isn't even that hard to prove, though we won't do it now.

The next question that arose (in a paper by Radner, 1972) was the following: what happens if there are securities that pay out in future contingencies (like stock in different companies), but not a complete set of Arrow-Debreu securities. This makes for arguably a more realistic model of actual securities markets. This question has given rise to a large "incomplete markets" literature in economics and finance. One of the interesting results from this literature is that without a complete set of A-D securities, the first welfare theorem generally won't hold. So there is potentially room for government intervention and policy questions become interesting.

10.2 A Simple Finance Model

In this section we introduce and study what is just about the simplest general equilibrium model with uncertainty. We assume there are two periods and in each state of the world there is just one consumption good. We normalize the spot price in each period to be equal to one.

We assume there are $S+1$ states of the economy. At time $t=0$ the economy is in state $s=0$; at time $t=1$ the economy can be in one of S possible states. In each state $s=0,...,S$ there is a single perishable consumption good.

Each agent $i \in \mathcal{I}$ has an initial endowment $e^i = (e^i_0, ..., e^i_S) \in \mathbb{R}^{S+1}_{++}$ and has a utility function $u^i : \mathbb{R}^{S+1}_{+} \to \mathbb{R}$ over consumption bundles $c^i = (c^i_0, ..., c^i_S) \in \mathbb{R}^{S+1}_{+}$. We asume each agent's utility function satisfies the standard assumptions — it's increasing, continuous and strictly concave. Also, let's define $\overline{x} = (x_1, ..., x_S)$ as the $t=1$ part of the vector $x = (x_0, x_1, ..., x_S)$. The aggregate endowment is $e = \sum_{i \in \mathcal{I}} e^i$.

There are J assets or securities. Each asset j pays dividends at date $t=1$ which we denote by $d^j \in \mathbb{R}^S$. The price of asset j at time $t=0$ is q_j. Without loss of generality we assume that these assets are in zero net supply (if we wanted the assets to be stock in some firm, there would be positive net supply, but then we could put the dividends into agent's endowments and be back to zero net supply). We collect all assets' dividends in the matrix:

$$A = (d^1, ..., d^J) \in \mathbb{R}^{S \times J}$$

At time $t=0$, each agent i chooses a portfolio $\alpha^i \in \mathbb{R}^J$, where α^i_j is the amount of asset j held by agent i. An agent's portfolio uniquely defines his wealth at each time one state, and hence his consumption (recall that prices are normalized to one at each date one state): $\overline{x}^i = e^i + A\alpha^i$ and $x^i_0 = e^i_0 - \alpha^i \cdot q$. The net demand of each agent $\overline{x}^i - e^i$ belongs to the span of the asset payoff matrix A :

$$\langle A \rangle = \{ z \in \mathbb{R}^S : \exists \alpha \in \mathbb{R}^J \text{ s.t. } z = A\alpha \}.$$

A finance economy is hence a triple: $\mathcal{E} = ((u^i, e^i)_{i \in \mathcal{I}}, A)$. Without loss of generality, we can assume that $\text{rank}(A) = J$ so there are no redundant assets. With redundant assets, an arbitrage argument would imply that the price of some assets would be uniquely determined by the price of other assets, regardless of preferences. We say that markets are *incomplete* if $J < S$.

Asset prices are said to be *arbitrage-free* if it is not possible to achieve a positive income stream in all states by trading at the going prices, i.e. if there is no position $\alpha \in \mathbb{R}^J$ with $q\alpha \leq 0$ and $A\alpha \geq 0$ with one inequality being strict. Here $q\alpha$ is the cost of portfolio α at date 0 and $A\alpha$ is the vector of payoffs at different date one staes. No arbitrage means you can't guarantee positive future income tomorrow without making a positive investment today.

If agents have strictly increasing utility functions, asset prices must preclude arbitrage or there would be a real problem with utility maximization. The absence of arbitrage is thus often seen *the* fundamental concept in finance (more so than equilibrium). Many important concepts (such as Black-Scholes option pricing) rely solely on arbitrage arguments.

Theorem 15 *An asset price vector $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a state price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi' \cdot A$.*

Proof. Let $M = \{(-q\alpha, A\alpha) : \alpha \in \mathbb{R}^J\}$ be the marketed subspace of $\mathbb{R}^{S+1}$. That is, $(-x_0, \overline{x}) \in M$ means that by spending x_0 at date 0, an agent can ensure the vector of returns $\overline{x}$ at date one. There is no arbitrage if and only if $\mathbb{R}^{S+1}_+ \cap M = \{0\}$. If $(x_0, \overline{x}) \in M$ and $x_0 \geq 0$, $\overline{x} \geq 0$ with a strict inequality (so $(x_0, \overline{x}) \in \mathbb{R}^{S+1}_+ - \{0\}$, it would be possible to start with zero wealth, consume x_0 today and consume $\overline{x}$ tomorrow — i.e. arbitrage would be possible.

For one direction of the proof, suppose there exists a strictly positive state price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi' A$. We show that this means there is no arbitrage. If there were also a vector $x \in \mathbb{R}^{S+1}_+ \cap M$ with $x \neq 0$, then because $x \in \mathbb{R}^{S+1}_+$ and $\pi \in \mathbb{R}^S_{++}$, we have $(1, \pi) \cdot x > 0$. But by the fact that $x \in M$ and $q = \pi' A$, we also have $(1, \pi) \cdot x = -q\alpha + q\alpha = 0$, a contradiction. Hence, a strictly positive state price vector implies no arbitrage.

For the converse direction, suppose no arbitrage: $\mathbb{R}^{S+1}_+ \cap M = \{0\}$. We use the separating hyperplane to derive a supporting state price vector. Note that M and

$\mathbb{R}_+^{S+1}$ are both convex sets whose intersection includes only the point $\{0\}$. The SHT asserts the existence of a vector $\mu \neq 0$ such that $\mu \cdot x < \mu \cdot z$ for all $x \in M$ and all non-zero $z \in \mathbb{R}_+^{S+1}$.[4]

Now, by the definition of M, it must be the case that if $x \in M$ then $-x \in M$,so we must have $\mu \cdot x = 0$ for all $x \in M$. Therefore $\mu \cdot z > 0$. The latter implies that $\mu \gg 0$ (if $\mu_l \leq 0$ for some l, we could find $z \in \mathbb{R}_+^{S+1} - \{0\}$ with $z_l > 0$ and $z_k = 0$ leading to the contradiction $\mu \cdot z \leq 0$). Therefore $-\mu_1 q + (\mu_2, ..., \mu_{S+1})A = 0$ and $\pi_s = \mu_{s+1}/\mu_1$ will give us a state price vector (note that to form π we just normalize the prices — μ has the right relative prices). *Q.E.D.*

A lot of asset pricing theory has to do with finding the right state-price vector π. Its existence is ensured by the absence of arbitrage, but often little can be said about it in general models.[5]

Definition 9 *A* ***financial markets equilibrium*** *for a finance economy* $\mathcal{E}$ *is a collection of portfolios* $\alpha^* = (\alpha^{1*}, ..., \alpha^{I*}) \in \mathbb{R}^{IJ}$*, individual consumptions* $(x^i)_{i \in \mathcal{I}}$ *and prices* $q^* \in \mathbb{R}^J$ *such that:*

1. *Agents maximize utility:*

$$(x^i, \alpha^{i*}) \in \quad \arg \max_{\alpha^i \in \mathbb{R}^J, c^i \in \mathbb{R}_+^{S+1}} u^i(c^i)$$
$$s.t. \; c^i = e^i + \begin{pmatrix} -q^{*\prime} \\ A \end{pmatrix} \alpha^i$$

2. *Markets clear:*

$$\sum_{i \in \mathcal{I}} \alpha^{i*} = 0$$

Clearly any equilibrium price vector must preclude arbitrage for the maximization problem to have a well-defined solution. Indeed, we can infer state prices from

[4] This is a slightly different version of the SHT than we used to prove the Second Welfare Theorem. There, we use the SHT to separate a convex set from a point outside that set. Here we are separating two disjoint convex sets M and $\mathbb{R}_+^{S+1} - \{0\}$. The idea is the same (you can check MWG's math appendix for a statement of both results).

[5] In dynamic models the state-price vector is sometimes called the pricing kernel, or the equivalent martingale measure (if normalized to add up to one).

the agents' first order conditions:

$$\pi_s = \frac{\partial u^i(x^i)}{\partial x^i_s}$$

If $J = S$ (remember the assets dividends are assumed to be linearly independent), then a financial markets equilibrium is equivalent to a Walrasian equilibrium. There will be a unique state-price vector $\pi \in \mathbb{R}^S_{++}$ such that $q = \pi' A$. This will be an equilibrium price vector for a Walrasian economy; the resulting allocations are the same in the two equilibria.

More interesting is the case where $J < S$ so that markets are incomplete. Under our assumptions, a financial markets equilibrium will still exist, but the equilibrium allocation may not be efficient. To see why, let's look at an example.

Suppose there are two states and there is a single bond that pays 1 in each state: $d = (1, 1)'$. Suppose there are two agents with endowments:

$$\begin{aligned} e^1 &= (1, 2, 1) \\ e^2 &= (1, 1, 2) \end{aligned}$$

and that both agents have identical utility:

$$u^i(x_0, x_1, x_2) = \log x_0 + \frac{1}{2} \log x_1 + \frac{1}{2} \log x_2$$

You can check as an exercise that the unique equilibrium will have no trade in the bond so everyone will just consume there endowment. This allocation, however, is Pareto dominated by the feasible allocation $x^1 = x^2 = (1, 1.5, 1.5)$.

The first welfare theorem fails because the set of existing securities does not allow the agents to suitably insure themselves against adverse states. There is still as sense, however, in which equilibrium exhausts the gains from trade.

Definition 10 *Given endowments $(e^i)_{i \in \mathcal{I}}$ and assets A, an allocation $(x^i)_{i \in \mathcal{I}}$ is* ***constrained efficient*** *if $\sum_{i \in \mathcal{I}} (x^i - e^i) \leq 0$, $x^i - e^i \in \langle A \rangle$ for all $i \in \mathcal{I}$ and there exists no alternative allocation $(\hat{x}^i)_{i \in \mathcal{I}}$ that Pareto dominates $(x^i)_{i \in \mathcal{I}}$ and also satisfies $\sum_{i \in \mathcal{I}} (\hat{x}^i - e^i) \leq 0$ and $\hat{x}^i - e^i \in \langle A \rangle$ for all $i \in \mathcal{I}$.*

If you're interested, you can try proving the following weaker welfare theorem:

Theorem 16 *If utility functions are strictly increasing, a financial markets equilibrium is constrained efficient.*

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