Optimal Auctions

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1 These slides are based on Paul Milgrom's.
What auction rules lead to the highest average prices?

**Difficulty:** the set of auction rules is enormous, e.g.:

- The auction is run in two stages. In stage one, bidders submit bids and each pays the average of its own bid and all lower bids.
- The highest and lowest first stage bid are excluded. All other bidders are asked to bid again.
- At the second stage, the item is awarded to a bidder with a probability proportional to the square of its second-stage bid. The winning bidder pays the average of its bid and the lowest second-stage bid.

Can we really optimize revenue over *all* mechanisms?
Revelation Principle

- For any augmented Bayesian mechanism \((\Sigma, \omega, \sigma)\), there is a corresponding direct mechanism \((T, \omega')\) for which truthful reporting is a Bayes-Nash equilibrium.
- The outcome function for the corresponding direct mechanism is given by:
  \[
  \omega' (t) = \omega (\sigma_1 (t_1), ..., \sigma_N (t_N)) .
  \]
Suppose that

- value of good to bidder $i$ is $v_i(t_i)$
- each $v_i$ is increasing and differentiable.
- types are independent, drawn from $U[0, 1]$.

**Definition**

An augmented mechanism $(\Sigma, \omega, \sigma)$ is **voluntary** if the maximal payoff $V_i(t_i)$ is non-negative everywhere.

The expected revenue from an augmented mechanism is the expected sum of payments:

$$R(\Sigma, \omega, \sigma) \triangleq \mathbb{E} \left[ \sum_{i=1}^{N} p_i^\omega(\sigma_1(t_1), \ldots, \sigma_N(t_N)) \right].$$
Revenue maximization problem

- Allow randomized mechanisms, and let $x_i(t)$ denote the probability that $i$ is awarded the good at equilibrium for the augmented mechanism.

- Then, we have the following constraints on feasible mechanisms:

$$x_i(t) \geq 0 \text{ for all } i \neq 0$$

$$\sum_{i=0}^{N} x_i(t) \leq 1$$

- Consider the problem

$$\max_{(\Sigma, \omega), \sigma \in NE} R(\Sigma, \omega, \sigma).$$

- Performance: $x_i(t_1, \ldots, t_N) \triangleq x_i^\omega(\sigma_1(t_1), \ldots, \sigma_N(t_N))$. 
Marginal revenue

- Simple case... assume there is a single bidder with value $v(t)$, where $v$ is increasing and $t$ is drawn from $U[0, 1]$.
- If the seller fixes a price $v(s)$ it sells whenever $t \geq s$, or with probability $1 - s$.
  - Can view $1 - s$ as the “quantity” sold
  - Expected total revenue is $(1 - s)v(s)$.
  - Marginal revenue is $d\text{Rev}/dQ$:

\[
MR(s) = v(s) - (1 - s)v'(s) = \frac{d((1 - s)v(s))}{d(1 - s)}
\]
Revenue formula

Theorem

Consider any augmented mechanism \((N, \Sigma, \omega, \sigma^*)\) for which \(\sigma^*\) is a BNE, \(x_j(t)\) is the probability that \(j\) is allocated the good at type profile \(t\), and \(V_j(0)\) is his expected payoff with type equal to zero. Then:

\[
R(\Sigma, \omega, \sigma^*) = \int_0^1 \cdots \int_0^1 \sum_{i=1}^N x_i(s_1, \ldots, s_N) MR_i(s) \, ds_1 \cdots ds_N - \sum_{i=1}^N V_i(0).
\]

Note that:

- Expected revenue is expressed as a function of the allocation \(x\) but not the payment function \(p\).
- The expression is *linear* in the \(x_i\)’s and depends critically on the *marginal revenues*.
Consider mechanism \((N, \Sigma, \omega)\) with equilibrium profile \(\sigma^*\).

Consider a given bidder \(i\). Its equilibrium payoff is:

\[
V_i(t_i) = \max_{\sigma_i \in \Sigma_i} \mathbb{E}_{t-i} \left[ x_i(\sigma_i, \sigma^*_{-i}(t-i)) v_i(t_i) - p_i(\sigma_i, \sigma^*_{-i}(t-i)) \right]
\]

\[
= \max_{\sigma_i \in \Sigma_i} \mathbb{E}_{t-i} \left[ x_i(\sigma_i, \sigma^*_{-i}(t-i)) \right] v_i(t_i) - \mathbb{E}_{t-i} \left[ p(\sigma_i, \sigma^*_{-i}(t-i)) \right]
\]

The derivative of this objective wrt \(t_i\) is:

\[
\mathbb{E}_{t-i} \left[ x_i(\sigma_i, \sigma^*_{-i}(t-i)) \right] v'_i(t_i)
\]

By the envelope theorem this expression gives \(V'_i(t)\)...
Derivation of revenue formula, II

- Applying Myerson’s Lemma, we have for any $i$,

$$V_i(t_i) - V_i(0) = \int_0^{t_i} \mathbb{E}_{s_{-i}} [x_i(s_i, s_{-i})] v_i'(s_i) ds_i$$

$$= \int_0^{t_i} \left( \int_0^1 \cdots \int_0^1 x_i(s_1, ..., s_N) ds_{-i} \right) v_i'(s_i) ds_i$$

- So the bidder’s average payoff across all types is:

$$\mathbb{E}_{t_i} [V_i(t_i)] - V_i(0) = \int_0^1 [V_i(t_i) - V_i(0)] dt_i$$
Using integration by parts we obtain

\[
\mathbb{E}_{t_i} [V_i(t_i)] - V_i(0) = \int_0^1 \left[ \int_0^{t_i} \mathbb{E}_{s_{-i}} [x_i(s_i, s_{-i})] \nu'_i(s_i) ds_i \right] dt_i
\]

\[
= \int_0^1 \mathbb{E}_{s_{-i}} [x_i(s_i, s_{-i})] \nu'_i(s_i) ds_i - \int_0^1 t_i \mathbb{E}_{s_{-i}} [x_i(t_i, s_{-i})] \nu'_i(t_i) dt_i
\]

\[
= \int_0^1 (1 - s_i) \mathbb{E}_{s_{-i}} [x_i(s_i, s_{-i})] \nu'_i(s_i) ds_i
\]

\[
= \int_0^1 \cdots \int_0^1 (1 - s_i) x_i(s_1, \ldots, s_N) \nu'_i(s_i) ds_1 \cdots ds_N
\]
Derivation of revenue formula, IV

- Total realized surplus is

\[ \sum_{i=1}^{N} x_i(t_1, \ldots, t_N)v_i(t_i) = x(t) \cdot v(t) \]

- Total expected revenue is therefore

\[
R(\Sigma, \omega, \sigma) = \mathbb{E}_t[x(t) \cdot v(t)] - \sum_{i=1}^{N} \mathbb{E}_{t_i}[V_i(t_i)]
\]

\[
= \int_0^1 \cdots \int_0^1 x_i(s_1, \ldots, s_N) \left[ v_i - (1 - s_i)v'_i(s_i) \right] ds_1 \ldots ds_N - \sum_{i=1}^{N} V_i(0)
\]

\[
= \int_0^1 \cdots \int_0^1 x_i(s_1, \ldots, s_N)MR_i(s_i) ds_1 \ldots ds_N - \sum_{i=1}^{N} V_i(0)
\]
Suppose that $MR_i$ is increasing for all $i$. Then and augmented voluntary mechanism is expected revenue maximizing if and only if (i) $V_i(0) = 0$ for all $i$, (ii) bidder $i$ is allocated the good exactly when $MR_i(t_i) > \max \{0, \max_{j \neq i} MR_j(t_j)\}$. Furthermore at least one such augmented mechanism exists, with expected revenue of:

$$E_t [\max \{0, MR_1(t_1), \ldots, MR_N(t_N)\}]$$.
From the prior theorem we have

\[ R(\Sigma, \omega, \sigma) = \int_0^1 \cdots \int_0^1 x_i(s_1, \ldots, s_N) MR_i(s_i) \, ds_1 \cdots ds_N - \sum_{i=1}^N V_i(0) \]

\[ \leq \int_0^1 \cdots \int_0^1 \max \{0, MR_1(s_1), \ldots, MR_N(s_N)\} \, ds_1 \cdots ds_N \]

The “0” in the max expression can be viewed as a “reserve price,” that is a condition under which the item is not sold.

We show on the next slide that the revenue bound can be achieved using a dominant strategy mechanism.
Ask bidders to report their types. Assign the item to the bidder with the highest marginal revenue, provided it exceeds zero.

Payments are

\[ p_i(t) = x_i(t) \cdot v_i \left( MR_i^{-1} \left( \max \left\{ 0, \max_{j \neq i} MR_j(t_j) \right\} \right) \right). \]

So bidder \( i \) pays only if she gets the item, and in that event she pays the value corresponding to the lowest type she could have reported and still been awarded the item.

It’s dominant to report one’s true type and \( V_i(0) = 0 \) for all \( i \).
An optimal auction may distort the allocation away from efficiency to increase revenue.

- Bidder 1’s value $v_1(t_1) = t_1$, so value is $U[0, 1]$.
- Bidder 2’s value $v_2(t_2) = 2t_2$, so value is $U[0, 2]$.

Marginal revenues:

$$MR_1(t_1) = t_1 - (1 - t_1) = 2t_1 - 1 = 2v_1 - 1$$
$$MR_2(t_2) = 2t_2 - 2(1 - t_2) = 4t_2 - 2 = 2v_2 - 2.$$

Bidder 1 may win despite having lower value, and auction may not be awarded despite both bidders having positive value.
Example, cont.

- Allocation in the optimal auction
The problem is analogous to standard multi-market monopoly problem.

- Each bidder is a separate “market” in which the good can be sold.
- Quantity is analogous to probability of winning – probabilities must sum to one like a quantity constraint.
- Monopolist can price discriminate, setting different prices in different markets.
- Allocating the probability of winning to individual markets is analogous to allocating quantities across markets.

Solution in both cases: sell marginal unit where marginal revenue is highest, provided it is positive!
Corollary

 Suppose valuation functions are identical $v_1 = \ldots = v_N = v$ and $v(s)$, $MR(s) = v(s) - (1 - s)v'(s)$ are increasing and take positive and negative values. A voluntary auction maximizes expected revenue iff at equilibrium, $V(0) = 0$, and the auction is assigned to the high value bidder provided its value exceeds $v(r)$, where $r$ satisfies $MR(r) = 0$. 
Corollary

Suppose the valuation functions are identical and $v(s)$, $MR(s)$ are increasing. Then the following auctions are optimal:

- A second price auction with minimum bid $v(r)$.
- A first price auction with minimum bid $v(r)$.
- An ascending auction with minimum bid $v(r)$.
- An all-pay auction with minimum bid $v(r) r^{N-1}$.
Suppose $N$ bidders and $v(t) = t$ for all bidders.

From above, $MR(t) = 2t - 1$.

The optimal reserve price is $v(r) = 1/2$, i.e. from $MR(r) = 0$.

In a second price auction, each bidder bids its value, but doesn’t bid if $t < 1/2$.

In a first price auction, each bidder places no bid if $t < 1/2$ and otherwise bids $\beta(t) = \left( \frac{N-1}{N} + \frac{(2t)^{-N}}{N} \right) t$. 

Bulow-Klemperer Theorem

Theorem

Suppose that the marginal revenue function is increasing and that \( v_i(0) = 0 \) for all \( i \). Then, adding a single buyer to an “otherwise optimal auction but with zero reserve” yields more expected revenue than setting the reserve optimally, that is,

\[
\mathbb{E} \left[ \max \{ MR_1(t_1), \ldots, MR_N(t_N), MR_{N+1}(t_{N+1}) \} \right] \\
\geq \mathbb{E} \left[ \max \{ MR_1(t_1), \ldots, MR_N(t_N), 0 \} \right].
\]
Lemma

\[ \mathbb{E}[MR_i(t_i)] = v_i(0) \]

**Proof.**

\[
TR_i(s) = (1 - s) v_i(s)
\]

\[
MR_i(s) = \frac{d}{d(1 - s)} TR_i(s) = -\frac{d}{ds} TR_i(s)
\]

so therefore

\[
\mathbb{E} [MR_i(s)] = \int_0^1 MR_i(s) ds = - [TR_i(1) - TR_i(0)] = v_i(0).
\]
Lemma

Given any random variable $X$ and any real number $\alpha$,

\[
\mathbb{E} \left[ \max\{X, \alpha\} \right] \geq \mathbb{E} [X] \quad \text{and} \\
\mathbb{E} \left[ \max\{X, \alpha\} \right] \geq \alpha, \quad \text{so} \\
\mathbb{E} \left[ \max\{X, \alpha\} \right] \geq \max \left\{ \mathbb{E} [X], \alpha \right\}.
\]

Proof. This follows from Jensen’s inequality.
Applying the two lemmata, with \( X = MR_{N+1}(t_{N+1}) \),

\[
\mathbb{E} \left[ \text{Revenue, } N + 1 \text{ bidders, no reserve} \right] \\
= \mathbb{E}_{t_1,...,t_N,t_{N+1}} \left[ \max \left\{ MR_1(t_1), ..., MR_N(t_N), MR_{N+1}(t_{N+1}) \right\} \right] \\
= \mathbb{E}_{t_1,...,t_N} \left[ \mathbb{E}_{t_{N+1}} \left[ \max \left\{ MR_1(t_1), ..., MR_N(t_N), MR_{N+1}(t_{N+1}) \right\} \right] \right] \\
\geq \mathbb{E}_t \left[ \max \left\{ MR_1(t_1), ..., MR_N(t_N), 0 \right\} \right] \\
= \mathbb{E} \left[ \text{Revenue, } N \text{ bidders, optimal auction} \right].
\]
Using the RET...

- Bulow-Klemperer theorem is one example of how to use the RET to derive new results in auction theory.
- Many other results also rely on RET arguments
  - McAfee-McMillan (1992) weak cartels theorem
  - Weber (1983) analysis of sequential auctions
  - Bulow-Klemperer (1994) analysis of dutch auctions
- Milgrom’s chapter five contains many examples.