Payoff Equivalence

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Market Design

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[1] These slides are based on Paul Milgrom's.

Payoff Equivalence: Motivation

- Debates in auction design frequently revolve around the revenue advantages of one format or another...
  - Advocates of sealed bid designs point to the "money left on the table" when the winner bids much more than the second highest bid ("doesn't happen in an ascending auction!").
  - Advocates of open ascending auctions point to bidders who are driven by competition to bid very near their maximum values ("doesn't happen in a sealed bid auction").
- Similar issues arise in debates about treasury auctions and electricity auctions (discriminatory vs uniform price).

Payoff Equivalence: Theory

- In early 1960s, Vickrey introduced the second price sealed bid auctions, and showed that it generated on average the same revenue as a first price sealed bid auction.
- In early 1980s, Myerson, Riley-Samuelson and Harris-Townsend explained this finding and generalized it to cover many “standard” auctions.
- A modern treatment emphasizes the envelope theorem.
- We have to start by reviewing Bayesian games and mechanisms.

Bayesian Mechanisms

- Players $1, \dots, N$
- Actions for player i denoted S_i.
- Outcome function $\omega : S_1 \times \dots \times S_N \to \Omega$.
- Also for each $i = 1, \dots, N$,
  - types $t_i \in T_i$
  - payoffs $u_i\left(\omega\left(s_1, \dots, s_N\right), t_1, \dots, t_N\right)$
  - beliefs $\pi_i\left(t_{-i} | t_i\right)$
  - strategies $\sigma_i : T_i \to S_i$.

Private Values

- Private values

$$u_i\left(\omega\left(s_1,...,s_N\right),t_1,...,t_N\right)=u_i\left(\omega\left(s_1,...,s_N\right),t_i\right).$$

- Interpretation: "I know my own preferences and values"
    - What if other players have relevant information (oil leases)?
    - What if a bidder may someday want to resell?
- General case is called "interdependent values."

Bayes-Nash Equilibrium

Definition

A strategy profile σ is a Bayes Nash equilibrium of Γ if for all players i and types t_i:

$$\begin{aligned}\sigma_i(t_i) &\in \arg\max_{s_i \in S_i} \mathbb{E}_i\left[u_i\left(\omega\left(s_i, \sigma_{-i}(t_{-i})\right), t_i, t_{-i}\right) \middle| t_i\right] \\ &= \arg\max_{s_i \in S_i} \int u_i\left(\omega\left(s_i, \sigma_{-i}(t_{-i})\right), t_i, t_{-i}\right) d\pi\left(t_{-i} \middle| t_i\right).\end{aligned}$$

Assumptions: homogenous good auctions

- Restrictive payoff assumption: $u_i = v_i(t_i)z_i(\sigma) - p_i$.
  - Quasi-linear payoffs
  - Risk neutrality
- Restrictive belief assumptions
  - Identical beliefs (symmetry)
  - Types independently distributed
- Additional assumptions
  - Payoff functions v_i continuously differentiable and non-decreasing in type.
  - Types distributed uniformly on $[0, 1]$ - nb doesn't imply values are uniform on $[0, 1]$.

Myerson's Lemma

- For any player i, independence of types implies

$$V_i\left(t_i;\sigma_{-i}\right)=\max_{s_i\in S_i}\mathbb{E}\left[z\left(s_i,\sigma_{-i}\left(t_{-i}\right)\right)v_i\left(t_i\right)-p_i\left(s_i,\sigma_{-i}\left(t_{-i}\right)\right)\right]$$

Lemma (Myerson)

Assume that types are independent and σ is a Bayes-Nash equilibrium. Let $z\left(\mathbf{t}\right)\triangleq z\left(\sigma_i\left(t\right),\sigma_{-i}\left(t_{-i}\right)\right)$ be the equilibrium probability vector for outcomes. Then i's expected payoff at type t_i is:

$$\begin{aligned}V_i\left(t_i;\sigma_{-i}\right) &= V_i\left(0,\sigma_{-i}\right)+\int_0^{t_i}\mathbb{E}_i\left[z_i\left(\mathbf{t}\right)|t_i=s\right]\cdot v_i'\left(s\right)ds\\ &= \mathbb{E}\left[z\left(\sigma_i\left(t\right),\sigma_{-i}\left(t_{-i}\right)\right)v_i\left(t_i\right)-p_i\left(\sigma_i\left(t\right),\sigma_{-i}\left(t_{-i}\right)\right)\right]\end{aligned}$$

Payoff & Revenue Equivalence

- The model is Vickrey's symmetric auction model with M indivisible goods for sale, identical independent atomless value distributions and each bidder able to buy just one item.
- Consider the class of mechanisms for which the equilibrium outcome is always efficient and the lowest type bidder always pays zero.

Theorem

In this model, for every such mechanism:

- *every type of every bidder has the same conditional expected payoff given its types as in the pivot mechanism.*
- *the seller's expected revenue is M times the expectation of the $M+1^{st}$ highest buyer value.*

Proof

- Let $z(\cdot)$ denote the efficient decision. In any auction satisfying the hypotheses, the expected payoff of a zero type bidder is zero. So by Myerson's lemma, the expected payoff of a type τ bidder is:

$$\int_0^t \mathbb{E}\left[z(\mathbf{t})\,|t_i = s\right] v_i'(s) ds$$

- The expected payoff to all parties, including the seller

$$\int_0^1 \cdots \int_0^1 \left(v\left(s^1\right) + \ldots + v\left(s^M\right)\right) ds_1 \cdots ds_N.$$

- Since the total payoff and bidders' average payoffs are the same as for the pivot mechanism, the seller's average payoff is also the same:

$$M \cdot \int_0^1 \cdots \int_0^1 v\left(s^{M+1}\right) ds_1 \cdots ds_N.$$

Examples

- M identical items for sale, N bidders want one each.
- Possible auction designs
  - Each of M highest bidders pays the $M + 1^{st}$ highest bid ("Vickrey auction" or "pivot mechanism").
  - Each of M highest bidders pays their own bid ("sealed bid auction").
  - Each of M highest bidders pays the lowest winning bid ("uniform price auction – T-bills).
- Surprising result: all lead to same average price!

Example: The Vickrey Auction

- Consider a bidder whose value is v and bids b.
- If M^{th} highest opposing bid if B, then bidder gets:
  - $v - B$ if $b \geq B$
  - 0 if $b < B$.
- Payoff is maximized by bidding $b = v$ regardless of what others bidders are doing.
- If all bidders play their dominant strategies, seller's revenue is M times $M + 1^{st}$ highest value.

Example: Sealed Tender

- M items, N bidders, minimum bid of zero.
- Each bidder has value v_i for an item, with v_i drawn
- Values are independently distributed.
- Auction rules: each bidder, knowing its value, places a bid. Top M bidders win and pay their bid.
- A strategy is a function $\beta : [0, V] \rightarrow \mathbb{R}_+$.
- If bid strategies are increasing, then

$$\beta(v) \in \arg\max_b (v - b) \Pr\left[b \geq \beta\left(v^{M:N-1}\right)\right].$$

Sealed Tender, foc approach

- First order condition for optimal bidding

$$0 = -\sum_{k=0}^{M-1} \frac{(N-1)!}{k!\,(N-1-k)!} \left(1 - F\left(\beta^{-1}(b)\right)\right)^k F^{N-1-k}\left(\beta^{-1}(b)\right) + \ldots$$

- At the solution, $b = \beta(v)$, so $\beta^{-1}(b) = v$ and f.o.c. becomes:

$$0 = -\sum_{k=0}^{M-1} \frac{(N-1)!}{k!\,(N-1-k)!} \left(1 - F(v)\right)^k F^{N-1-k}(v) + (v - \beta(b)) \ldots$$

- Re-arrange to get "envelope" formula:

$$\frac{d}{dv}\left((v - \beta(v)) \sum_{k=0}^{M-1} \frac{(N-1)!}{k!\,(N-1-k)!} \left(1 - F(v)\right)^k F^{N-1-k}(v)\right)$$

$$= \sum_{k=0}^{M-1} \frac{(N-1)!}{k!\,(N-1-k)!} \left(1 - F(v)\right)^k F^{N-1-k}(v)$$

Sealed tender, foc approach, cont.

- Integrating up and re-arranging

$$\beta(v) = v - \frac{K + \int_0^v \sum_{k=0}^{M-1} \frac{(N-1)!}{k!(N-1-k)!} (1 - F(s))^k F^{N-1-k}(s)\, ds}{\sum_{k=0}^{M-1} \frac{(N-1)!}{k!(N-1-k)!} (1 - F(v))^k F^{N-1-k}(v)}$$

- If minimum bid is zero, so is the constant of integration K (why?).

Sealed tender, RET approach

- Can also "guess" the equilibrium by equating our expression for equilibrium profits with the envelope theorem formula (using the fact that lowest-value type gets zero):

$$
\begin{aligned}
&(v-\beta(v))\sum_{k=0}^{M-1}\frac{(N-1)!}{k!\,(N-1-k)!}\left(1-F\left(v\right)\right)^{k}F^{N-1-k}\left(v\right)\\
=\ &\int_0^v\sum_{k=0}^{M-1}\frac{(N-1)!}{k!\,(N-1-k)!}\left(1-F\left(s\right)\right)^{k}F^{N-1-k}\left(s\right)ds
\end{aligned}
$$

- This is the slick way to solve many auction models.

Importance of Revenue Equivalence Theorem

- Result is striking but it relies on restrictive assumptions that often do not hold in practice.
- Many important results in economics fall into this category:
  - First and second welfare theorem
  - Coase theorem
  - Modigliani-Miller theorem
  - Revenue equivalence theorem
- Why are these kinds of results significant?

Bilateral Bargaining

- Seller with production cost c, buyer with value b both drawn from $[0, 1]$.
- In the pivot mechanism, parties trade iff $b > c$.
  - Total gains from trade (surplus): $\max\{0, b - c\}$.
  - If trade takes place,
    - Seller receives price b and earns $\max\{b - c, 0\}$.
    - Buyer pays price c and earns $\max\{b - c, 0\}$.
- On average, the VCG mechanism incurs a loss equal to the average gains from trade, which is $\mathbb{E}[\max\{b - s, 0\}]$.

Bilateral Bargaining, cont.

- The outcome of bargaining may depend on the bargaining protocol; we can view protocols as *mechanisms* used for bargaining.
- The pivot mechanism supports efficient outcomes and requires no payment if there is no trade, but requires a (huge) subsidy equal to the total surplus.
- Question
  - Is there any mechanism that leads to efficient outcomes at a Bayes-Nash equilibrium and requires no payment if there is no trade and requires a smaller average subsidy?

Myerson-Satterthwaite Theorem

Theorem (Myerson-Satterthwaite)

Any mechanism that

1. *results in efficient trade in the two-person bargaining problem at Bayes-Nash equilibrium, and*
2. *entails no payments when there is no trade*

incurs an expected loss for the mechanism operator equal to $\mathbb{E}[\max\{0, b-c\}]$, *which is also the expected gains from trade.*

- Proof.
  - By Myerson's lemma, any mechanism that implements efficient trade with $V_b(0) = 0$ and $V_s(1) = 0$ has the same expected payoff for each type of the buyer and each type of the seller as the pivot mechanism.
  - The expected total surplus is $\mathbb{E}[\max\{0, b-c\}]$ and each player expects that same payoff....

FCC Auction Debate

- In a famous article from many years ago, Coase (1959) argued that the FCC should auction spectrum licenses. He pointed out that in a world of no "transaction costs," the assignment of ownership wouldn't matter, but the real world wasn't like that....
- In the early 1990s, the FCC started to think seriously about using auctions. There was a debate about whether the form of auction would matter at all for eventual efficiency.
- Winning argument at the time (made by Milgrom)....
    - In theory efficient allocations are implementable in a private value auction environment (VCG result).
    - In theory efficient allocations are not implementable if we start by randomly assigning the licenses (Myerson-Satterthwaite).