Quantum Ergodicity for a Class of Mixed Systems

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Let $(M, g)$ be a smooth compact manifold with or without boundary. We wish to examine the behavior of high energy eigenfunctions of $-\Delta_g$.

- Do eigenfunctions concentrate on subsets of the manifold or spread out uniformly?
- Is this behavior related to the classical dynamics of the geodesic flow on $M$? e.g. ergodicity, complete integrability
Billiards Examples

[Diagram of a billiard table with a ball in motion]

Quantum Ergodicity Mixed Systems
Billiards Examples
Basic Quantum Ergodicity Theorem

Theorem (Shnirelman ’74, Colin de Verdière ’85, Zelditch ’87, Zelditch-Zworski ’96)

Let \((M, g)\) be a compact manifold with piecewise smooth boundary such that the geodesic flow on \(S^*M\) is ergodic. Let \(\psi_j, j = 1, \ldots\) be the eigenfunctions of \(-\Delta_g\). Then, there is a full density subsequence of eigenfunctions, \(\psi_{jk}\), such that

\[ \psi_{jk} \rightharpoonup 1. \]
High Energy Eigenfunction
Phase Space?

Ergodic flow actually spreads uniformly in phase space (on $S^*M$).

- Can we get a similar "uniform spreading" for eigenfunctions in phase space?
- How can we quantify such notions?
**Definition**

$A \in \Psi^m$ is a pseudodifferential operator of order $m$ if

$$A(x, D)u = \frac{1}{(2\pi)^n} \int \int e^{i\langle x-y, \xi \rangle} a(x, \xi) u(y) dyd\xi$$

for some $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|}.$$

$a$ is called the symbol of $A$ ($\sigma(A)$).

We say $A$ is homogeneous of degree $m$ if $\sigma(A)$ is homogeneous of degree $m$ in the $\xi$ variables.

**Remark:** We define similar operators use local coordinate charts on manifolds.
Defect Measures

Definition

Let $\psi_j, j = 1, \ldots$ be a sequence of functions in $L^2$. Then, a defect measure, $\mu$, for $\psi_j$ is a Radon measure on $S^*M$ such that for all $A \in \Psi^0$, homogeneous pseudodifferential operators of order 0, one has

$$\langle A\psi_j, \psi_j \rangle \to \int_{S^*M} \sigma(A) d\mu.$$

Theorem

Let $\{\psi_j, j = 1, \ldots\}$ have $\sup_j \|\psi_j\| < \infty$. Then there exists a subsequence $\psi_{j_k}$ with a defect measure $\mu$. 
Quantum Ergodicity Theorem

**Theorem (Shnirelman ’74, Colin de Verdière ’85, Zelditch ’87, Zelditch-Zworski ’96)**

Let $(M, g)$ be a compact manifold with boundary such that the geodesic flow on $S^*M$ is ergodic. Let $\psi_j$, $j = 1, ...$ be the eigenfunctions of $-\Delta_g$. Then, there is a full density subsequence of eigenfunctions, $\psi_{j_k}$, such that $\psi_{j_k}$ has a defect measure, $\mu$, and $\mu \equiv \mu_L$ where $\mu_L$ is the normalized Liouville measure on $S^*M$. 
Other Dynamical Assumptions

We have an understanding of what happens when the flow is ergodic, but what happens when the geodesic flow is not completely ergodic?

- Completely integrable?
- "Almost ergodic" when considered on only physical space?
  - Marklof-Rudnick (2012) addresses this case.
- What if phase space is divided? i.e. has an invariant subset on which the flow is ergodic?
  - Can we get a similar quantum ergodicity result?
  - Percival (1973) conjectured that high energy eigenfunctions should follow the quantum classical correspondence principle and hence that eigenfunctions should spread uniformly in areas of ergodicity.
Main Theorem: Divided Phase Space

Theorem (G. 2012)

Let $(M, g)$ be a compact manifold with a piecewise smooth boundary. Let $\varphi_t$ be the geodesic flow and suppose that there exists $U \subset S^* M$, $\mu_L(U) > 0$, $\mu_L(\partial U \setminus U) = 0$, $\varphi_t$ is ergodic on $U$, $\varphi_t(U) = U$.

Let $\{\psi_j, j = 1, \ldots\}$ be the eigenfunctions for $-\Delta_g$. Then there is a full density subsequence of $\psi_j$ such that if any further subsequence $\psi_{j_k}$ has defect measure $\mu$, then $\mu|_U = c\mu_L|_U$ for some $c$.

Remark: This result can be generalized to a semiclassical setting and to almost orthogonal quasimodes rather than eigenfunctions.
Prototype Example: Mushroom Billiards
Eigenfunctions on Mushroom Billiards
Ingredients in Proof

1. Proof of classical quantum ergodicity theorem
   - Weak fixed time Egorov’s Theorem
   - Local Weyl Law

2. Basic measure theory