REVIEW ON FOURIER ANALYSIS AND SOBOLEV THEORY

(1) Given a function \( f \in L^1(\mathbb{R}^n) \), define the Fourier transform by the formula
\[
\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx.
\]
Show that if \( f(x) = e^{-\pi a |x|^2} \) where \( a > 0 \), then \( \hat{f}(\xi) = a^{-\frac{n}{2}} e^{-\frac{\pi |\xi|^2}{a}} \). (Hint: Start with \( n = 1 \). First prove that if \( x^\alpha h \in L^1 \), then
\[
\partial^\alpha \hat{h} = [(2\pi i x)^\alpha h] .
\]
Using this identity, prove that for \( f \) defined above, \( e^{\pi \xi^2} \hat{f}(\xi) \) is independent of \( \xi \). Then conclude the \( n = 1 \) case using the fact that \( \int e^{-\pi ax^2} \, dx = a^{-\frac{1}{2}} \).

(2) (Fourier inversion formula) Define the inverse Fourier transform by
\[
\check{g}(x) := \int_{\mathbb{R}^n} \hat{g}(\xi) e^{2\pi i x \cdot \xi} \, d\xi.
\]
In this exercise, follow the steps below to show that this name is justified, i.e., for \( f, \hat{f} \in L^1(\mathbb{R}^n) \), we have
\[
(\hat{\check{f}}) = f \tag{1}
\]
almost everywhere.

(a) First show that
\[
\int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} f(x) \check{g}(x) dx .
\]
(b) Show that
\[
(\hat{\check{f}})(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} e^{-\epsilon^2 |\xi|^2 + 2\pi i x \cdot \xi} \hat{f}(\xi) d\xi .
\]
(c) Let \( g(\xi) = e^{-\epsilon^2 |\xi|^2 + 2\pi i x \cdot \xi} \). Show using problem 1 that
\[
\check{g}(y) = e^{-\epsilon^2 |x-y|^2} .
\]
(d) Let \( \phi \) be a smooth function such that \( \int_{\mathbb{R}^n} \phi = 1 \). Show that if \( \beta \) is an \( L^p \) function, then
\[
\int_{\mathbb{R}^n} \beta(\phi(y) e^{-\gamma \phi(\frac{y-x}{\epsilon})} dy \) converges to \( \beta(x) \) in \( L^p \) as \( \epsilon \to 0 \).

(e) Combine the previous parts and conclude (1) using
\[
\int_{\mathbb{R}^n} e^{-\pi |x|^2} \, dx = 1 .
\]

(3) (Plancherel’s theorem) Let \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). Show that
\[
\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} .
\]
(Hint: Use 2a.)

(4) Show that the Fourier transform maps the Schwartz class into itself. Here we recall that the Schwartz class is defined as
\[
\mathcal{S} := \{ f \in C^\infty : \sup_x |x^\alpha \partial_x^\beta f| \leq C_{\alpha,\beta} \text{ for all multi-indices } \alpha, \beta \} .
\]
Here, we have used the multi-index notation, i.e., for \( \alpha = (\alpha_1, \alpha_2, ... \alpha_n) \), we denote by \( x^\alpha \) the function \( x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n} \) and \( \partial_x^\alpha \) the differential operator \( \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} \). (Hint: Use the formal identity
\[
(x^\alpha \partial_x^\beta f) = (-1)^{|\alpha|} (2\pi i)^{|\beta|-|\alpha|} \partial_\xi^\beta (\xi^\beta f) .
\]
(5) Define the function space $H^s(\mathbb{R}^n)$ as the completion of $C^\infty_c(\mathbb{R}^n)$ under the norm
\[
\|f\|_{H^s(\mathbb{R}^n)} := \|\hat{\xi}^s \hat{f}\|_{L^2(\mathbb{R}^n)}.
\]
First, show that this is indeed a norm. Then, show that when $s$ is a non-negative integer, there exists a constant $C$ (depending only on $s$) such that
\[
C^{-1} \sum_{|\alpha| = s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C \sum_{|\alpha| = s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.
\]
(Here, we have used the multi-index notation as before). Also, define $H^s$ as the completion of $C^\infty(\mathbb{R}^n)$ under the norm
\[
\|f\|_{H^s(\mathbb{R}^n)} := \|\hat{(1 + |\xi|)^s \hat{f}}\|_{L^2(\mathbb{R}^n)}
\]
and show that if $s$ is a non-negative integer, we have
\[
C^{-1} \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.
\]
(6) (Hausdorff-Young) Prove that for $p \geq 2$
\[
\|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^{p'}(\mathbb{R}^n)},
\]
where $1 \leq p' \leq 2$ is defined via the relation $\frac{1}{p} + \frac{1}{p'} = 1$. (Hint: Consider the case $p = \infty$. Then use Plancheral’s theorem together with Reisz-Thorin interpolation theorem.)

(7) (Sobolev embedding theorem I) Prove that there exists a constant $C = C(n, s) > 0$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$, we have
\[
\|f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{H^s(\mathbb{R}^n)}.
\]
(Hint: Use the Hausdorff-Young inequality and the Plancheral theorem.)

(8) (Sobolev embedding theorem II) Show that when $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$ and $p < \infty$, there exists a constant $C = C(n, p) > 0$ such that
\[
\|f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{H^s(\mathbb{R}^n)}.
\]
(Hint: Use the Hardy-Littlewood-Sobolev inequality:
\[
\|f \ast \frac{1}{|x|^\alpha}\|_{L^s(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}
\]
for $\frac{1}{p} = \frac{1}{q} + \frac{2 - \alpha}{n}$, $1 < p < q < \infty$, $0 < \alpha < n$. )