Autumn 2016 Math 175: Homework Assignment 1 (Due 10/7 4pm)

1. (10 points) Problem 1.4
2. (10 points) Problem 1.10
3. (10 points) Problem 2.7 (cf. Problem 2.1)
4. (10 points) Problem 2.8
5. (10 points) Problem 2.13
6. (30 points) Let \((V, \| \cdot \|)\) be a normed vector space over \(\mathbb{R}\) such that the parallelogram law holds, i.e., for every \(a, b \in V\),

\[
2\|a\|^2 + 2\|b\|^2 = \|a + b\|^2 + \|a - b\|^2.
\]

Define the map \(\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}\) by

\[
\langle x, y \rangle := \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right).
\]

The goal of this problem is to show that \(\langle \cdot , \cdot \rangle\) defines an inner product.

(a) First, for every \(x, y, z \in V\), show that \(\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle\). (Hint: Apply the parallelogram law with \((a, b) = (x+z, y), (a, b) = (y+z, x), (a, b) = (x - z, y), (a, b) = (y - z, x)\).)

(b) Then, using the previous step or otherwise, show that for every \(x, y \in V\) and for every \(n \in \mathbb{N}\),

\[
\langle nx, y \rangle = n\langle x, y \rangle.
\]

(c) Next, using the previous step or otherwise, show that for every \(x, y \in V\) and for every \(q \in \mathbb{Q}\),

\[
\langle qx, y \rangle = q\langle x, y \rangle.
\]

(d) Show that \(\langle \cdot , \cdot \rangle\) is linear in the first argument.

(e) Conclude that \(\langle \cdot , \cdot \rangle\) is an inner product.