Autumn 2016 Math 175: Homework Assignment 6 (Due 11/11 4pm)

1. (10 points) Problem 7.1
2. (10 points) Problem 7.9
3. (10 points) Problem 7.10
4. (10 points) Problem 7.11
5. (10 points) Problem 7.20

6. (20 points) In this problem, we define the adjoint operator in a more general setting, which applies for operators on Banach spaces which are not necessarily Hilbert spaces. Let $X, Y$ be Banach spaces over $\mathbb{C}$ and $T \in L(X, Y)$. For every $f \in Y^*$, define $\bar{T}^*(f) : X \to \mathbb{C}$ by

$$\bar{T}^*(f)(x) = f(Tx)$$

for all $x \in X$.

(a) Prove that $\bar{T}^* \in L(Y^*, X^*)$.
(b) Prove that $\|\bar{T}^*\|_{L(Y^*, X^*)} \leq \|T\|_{L(X, Y)}$.
(c) In the case $X, Y$ are Hilbert spaces, show that for $T^* : Y \to X$ being the adjoint according to the definition in class (i.e., $\langle Tv, w \rangle_Y = \langle v, T^*w \rangle_X$ for $v \in X, w \in Y$) and $\phi_X : X \to X^*$ and $\phi_Y : Y \to Y^*$ being the isometric conjugate-isomorphisms $\phi_X(x_1)(x_2) = \langle x_2, x_1 \rangle$ and $\phi_Y(y_1)(y_2) = \langle y_2, y_1 \rangle$ ($x_1, x_2 \in X, y_1, y_2 \in Y$) in the Riesz–Fréchet representation theorem, we have

$$\phi_X^{-1} \circ \bar{T}^* \circ \phi_Y = T^*.$$

7. (20 points) Let $H$ be a Hilbert space, we say that $T \in L(H)$ is normal if $TT^* = T^*T$. Prove that if $T$ is normal, then $\|Tx\|_H = \|T^*x\|_H$ for every $x \in H$. Conclude that $\ker(T) = \ker(T^*) = \text{im}(T)^\perp = \text{im}(T^*)^\perp$. 