1. (I.4) Let \( \{x_n\}_{n=1}^{\infty} \) be a Cauchy sequence in a metric space \((X, d)\). Suppose that for some subsequence \( \{x_{n(i)}\}_{i=1}^{\infty} \), \( x_{n(i)} \to x_{\infty} \) as \( i \to \infty \). Prove that \( x_n \to x_{\infty} \).

2. (I.5) Prove the following theorem: If \((M, d)\) is an incomplete metric space, it is possible to find a complete metric space \(\tilde{M}\) so that \(M\) is isometric to a dense subset of \(\tilde{M}\). (Note that the textbook gives a sketch of the proof — complete it!)

3. (III.4) Show that all norms on \(\mathbb{R}^n\) are equivalent. (Hint: Use the fact that the unit sphere is compact in the Euclidean topology.)

4. (I.7) Suppose that \(T\) is a linear transformation between two normed vector spaces. Show that the following are equivalent:
   
   (a) \(T\) is continuous at one point,
   
   (b) \(T\) is continuous at all points,
   
   (c) \(T\) is uniformly continuous,
   
   (d) \(T\) is bounded.

5. (I.32) Let \(F : C([0, 1] \times [0, 1])\) and consider the map \(\mathcal{F} : C([0, 1]) \to C([0, 1])\) given by
   
   \[(\mathcal{F}f)(x) = \int_0^1 F(x, y)f(y)dy.\]
   
   Prove that \(\{\mathcal{F}f : \|f\|_{L^\infty} \leq 1\}\) is an equicontinuous family so that any given sequence \(\{f_n\}_{n=1}^{\infty}\) with \(\|f_n\| \leq 1\) for all \(n\) has a subsequence \(f_{n(i)}\) uniformly convergent.

6. (II.6) Let \(M\) be a linear subspace of a Hilbert space \(H\). Prove that \(M^\perp\) is a closed linear subspace and \(\overline{M} = (M^\perp)^\perp\), with the bar denoting closure.

7. (II.9) Let \(M\) be a subspace of a Hilbert space \(H\). Let \(f : M \to \mathbb{C}\) be a linear functional on \(M\) with norm \(C\).
   
   Prove that there is a unique extension of \(f\) to a linear functional \(\tilde{f} : H \to \mathbb{C}\) with the same norm.

   (Note that the existence part would follow from the Hahn–Banach theorem (which will be covered on Week 3), but the point of this problem is that it is not necessary in the Hilbert space setting.)