Winter 2018 Math 205B: Homework
Assignment 5 (Due 2/14 during lecture)

1. Reed–Simon IV.23.
2. Reed–Simon IV.30.
3. Reed–Simon IV.36.
4. Reed–Simon IV.40.

5. Suppose that $X$ is a Banach space over $F = \mathbb{C}$ or $F = \mathbb{R}$, and $\tau$ is its weak topology.

(a) Suppose that $f \in X^*$, $f_1, \ldots, f_n \in X^*$, and there is a constant $C > 0$ such that for $x \in X$, $|f_j(x)| < C$ for $j = 1, 2, \ldots, n$ implies $|f(x)| < 1$. Show that $\bigcap_{j=1}^n \ker(f_j) \subset \ker(f)$. Use this to conclude that $f$ is a linear combination of the $f_j$.

(b) Suppose that $(X, \tau)$ is first countable. Show that there are linear functionals $f_1, f_2, \ldots \in X^*$, such that every $f \in X^*$ is a finite linear combination of the $f_j$. That is, if $f \in X^*$ then there exists $N > 0$ and $a_j \in F$, $j = 1, 2, \ldots, N$ such that $f = \sum_{j=1}^N a_j f_j$. (Hint: Suppose $B = \{B_1, B_2, \ldots\}$ is a (countable) base at 0. Show that you may assume that, i.e. that there exists another base at 0 such that, there are linear functionals $f_j \in X^*$, $j = 1, 2, \ldots$ and a monotone decreasing sequence $\{\epsilon_j\}_{j=1}^\infty$ with $B_n = \{x \in X : |f_j(x)| < \epsilon_n, j = 1, 2, \ldots, n\}$. Now use that $f \in X^*$ is continuous also from $(X, \tau)$ to $F$.)

(c) Suppose that $X$ is a separable infinite dimensional Hilbert space, and let $\tau$ denote its weak topology. Show that $(X, \tau)$ is not first countable, and hence the topology is not induced by a metric.

6. Suppose that $X$ is a Banach space. Let $\tau$ denote the weak topology on $X$.

(a) Show that $(X, \tau)$ is (T3), i.e. (cf. Definition on p.94 in Reed–Simon) every point is closed, and given any $x \in X$ and $C \subset X$ closed with $x \notin C$, there are open sets $U_1, U_2$ such that $x \in U_1$, $C \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

(b) Show that $(X, \tau)$ has the following property, sometimes called (T3\(\frac{1}{2}\)) or completely regular: every point is closed, and given any $x \in X$ and $C \subset X$ closed with $x \notin C$, there is a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and $f$ identically 0 on $C$. (Note: $f^{-1}((\frac{1}{2}, 1])$ and $f^{-1}([0, \frac{1}{2}])$ are then disjoint open sets, so this actually implies (T3).)