1. Reed–Simon IV.19

2. Reed–Simon IV.20

3. Suppose that $(X, d)$ is a metric space.
   (a) Suppose that $f : [0, \infty) \to [0, \infty)$ satisfies $f(0) = 0$, $f(x) > 0$ if $x > 0$, $f$ is increasing (i.e. $x \geq y$ implies $f(x) \geq f(y)$) and $f$ is subadditive: $f(x + y) \leq f(x) + f(y)$ for all $x, y$. Show that $f \circ d : X \times X \to [0, \infty)$ is a metric on $X$.
   (b) Suppose that $f : [0, \infty) \to [0, \infty)$ is $C^1$ (continuously differentiable), $f(0) = 0$, $f'(0) > 0$, $f'(x) \geq 0$ for all $x$, and $f'$ is decreasing (i.e. $x \leq y$ implies $f'(x) \geq f'(y)$). Show that $f$ is subadditive.
   (c) Suppose $d$ and $d'$ are metrics on $X$. Show that the topology generated by $d'$ is weaker than the topology generated by $d$ (i.e. every open set in $(X, d')$ is open in $(X, d)$) if and only if given $\epsilon > 0$ and $x \in X$ there is $\delta > 0$ such that
   \[ d(x, y) < \delta \implies d'(x, y) < \epsilon. \]
   Use this to show that with $f$ as in part (b), $d$ and $f \circ d$ generate the same topology. (Such metrics are called equivalent.)
   (d) Conclude that if $d$ is a metric on $X$, then so are $d' = \frac{d}{1+d}$ and $d'' = \min\{d, 1\}$, and these metrics generate the same topology. (Note that $d'(x, y) < 1$ for all $x, y \in X$, and $d''(x, y) \leq 1$ for all $x, y \in X$.)

4. A pseudometric $\rho$ on a set $X$ is a map $\rho : X \times X \to [0, \infty]$ that is symmetric, satisfies the triangle inequality and $\rho(x, x) = 0$ for all $x \in X$. (So if $\rho$ is a pseudometric and $\rho(x, y) = 0$ implies $x = y$ then $\rho$ is a metric.) Let $\rho_1, \rho_2, \ldots$, be pseudometrics on $X$ with $\rho_j \leq 1$. Let
   \[ d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y). \]  
   (1)

   (a) Show that $d$ is a pseudometric on $X$.
   (b) Show that if $x \neq y$ implies that $\rho_j(x, y) \neq 0$ for some $j$ then $d$ is a metric on $X$, and a sequence $\{x_n\}_{n=1}^{\infty}$ converges to some $x \in X$ with respect to $d$ if and only if given $\epsilon > 0$ there is $N$ such that $n \geq N$ implies $\rho_j(x_n, x) < \epsilon$. Thus, if the $\rho_j$ are metrics, then a sequence converges with respect to $d$ if and only if it converges with respect to $\rho_j$ for every $j$.
   (c) Show that the topology generated by $d$ is the weak topology generated by $\{\rho_n(x, \cdot) : x \in X, n \in \mathbb{N}\}$, i.e. the weakest topology in which these functions are all continuous.
5. Suppose that \( X \) is a vector space and each \( d_j \) is a translation invariant metric, i.e. \( d_j(x + z, y + z) = d_j(x, y) \) for all \( x, y, z \in X \). Let \( \rho_j \) be translation invariant metrics equivalent to \( d_j \) with \( \rho_j \leq 1 \); and let \( d \) be defined by (1). Show that a sequence \( \{x_n\}_{n=1}^\infty \) is Cauchy with respect to \( d \) if and only if it is Cauchy with respect to every \( d_j \).

(e) Now suppose that \( X_1, X_2, \ldots \) are vector spaces, \( X_1 \supseteq X_2 \supseteq \ldots \) and \( X = \cap_{k=1}^\infty X_k \). Let \( d_k \) be translation invariant metrics on \( X_k \), and suppose that the inclusion maps \( \iota_k : X_k \to X_{k-1} \) are all continuous. Let \( d \) be defined as in the previous part. Show that if \( (X_k, d_k) \) is complete for every \( k \) then \( (X, d) \) is complete.

(f) Let \( C^\infty(S^1) \) denote the set of complex-valued infinitely differentiable functions on \( S^1 = \mathbb{R}/(2\pi \mathbb{Z}) \). Let \( d_k \) be the metric given by the \( C^k \) norm:

\[
\|f\|_{C^k} = \sum_{m=0}^k \sup \{|f^{(m)}(x)| : x \in S^1\}.
\]

Let \( d \) be the corresponding metric (defined as in part (d)) on \( C^\infty(S^1) \). Show that \( C^\infty(S^1) \) is a complete metric space in which sequences \( \{x_n\}_{n=1}^\infty \) converge, resp. are Cauchy, if and only if they converge, resp. are Cauchy, in every \( C^k \). (Thus, convergence of a sequence \( \{f_n\}_{n=1}^\infty \) is just the uniform convergence of all derivatives \( \{f_n^{(k)}\}_{k=1}^\infty \).)

5. Suppose that \( (X, \tau) \) is a compact topological space and let \( \mathcal{F} = \{f_1, f_2, \ldots\} \) be a countable collection of continuous real valued functions on \( X \) that separate points (i.e. for \( x \neq y \) there is a \( j \) such that \( f_j(x) \neq f_j(y) \)).

(a) Show that without loss of generality we may assume \( |f_j(x)| < \frac{1}{2} \) for all \( x \); assume this from now on.

(b) Let \( \rho_j(x, y) = |f_j(x) - f_j(y)| \). Show that \( \rho_j \) is a pseudometric. With \( d \) given by Equation (1) of Problem 4, show that \( d \) is a metric on \( X \); let \( \tau_d \) denote the metric topology on \( X \).

(c) Show that \( \tau_d = \tau \), i.e. \( (X, \tau) \) is metrizable. (Hint: show that \( \tau_d \) is Hausdorff and it is the \( \mathcal{F} \)-weak topology, hence is weaker than \( \tau \).)

(d) Suppose \( Y \) is a separable Banach space. Show that the closed unit ball \( B \) in \( Y^* \) is metrizable in the weak-* topology. Conclude the following version of the sequential Banach–Alaoglu theorem: If \( Y \) is a separable Banach space, then the closed unit ball \( B \) in \( Y^* \) is sequentially compact in the weak-* topology, i.e. for any sequence in \( B \), there exists a convergent subsequence in the weak-* topology. (Note that by Problem 5 in HW 5, in general \( Y^* \) itself is not metrizable in the weak-* topology.)

(e) Suppose \( \{u_n\}_{n=1}^\infty \subset L^2([0,1]) \) satisfies \( \sup_{n \in \mathbb{N}} \|u_n\|_L^2([0,1]) \leq 1 \). Prove that there is a subsequence \( \{u_{n(m)}\}_{m=1}^\infty \) and an \( u \in L^2([0,1]) \) such that \( \int_0^1 u_{n(m)}(x)v(x) \, dx \to \int_0^1 u(x)v(x) \, dx \) as \( m \to +\infty \) for every \( v \in L^2([0,1]) \).
(f) (See also Reed–Simon IV.20) Let $Y = \ell^\infty$ and $\delta_1, \delta_2, \cdots \in Y^*$ be defined by

$$
\delta_n(\{c_k\}_{k=1}^\infty) = c_n.
$$

Prove that $\{\delta_n\}_{n=1}^\infty$ has no weak-* convergent subsequence. (This shows that for a general non-separable Banach space $Y$, the closed unit ball $B$ in $Y^*$ may not be sequentially compact.)