Winter 2018 Math 205B: Midterm I

This is a closed-book, closed-notes, closed-phone, etc., exam. Answer all questions. You have 50 minutes.

1. Let \((X, \| \cdot \|_X)\) be a Banach space and \(Y\) be a closed subspace. Suppose there is a complete norm \(\| \cdot \|_Y\) on \(Y\) such that the identity map \(\text{Id} : (Y, \| \cdot \|_Y) \to (Y, \| \cdot \|_X)\) is continuous.

(a) Prove that \(\| \cdot \|_X\) and \(\| \cdot \|_Y\) are equivalent norms on \(Y\).

(b) Give an example to show that the closedness of \(Y\) is a necessary condition for the result in part (a) to hold.

2. Let \(\ell^\infty := \{(a_1, a_2, \ldots) : a_i \in \mathbb{R}, \forall i \in \mathbb{N}, \sup_i |a_i| < +\infty\}\) with the usual \(\ell^\infty\) norm.

(a) Prove that there exists \(\lambda \in (\ell^\infty)^*\) such that

\[
\lambda(a) \leq \limsup_{i \to \infty} a_i,
\]

(b) Using the previous part, or otherwise, show that there exists \(\lambda \in (\ell^\infty)^*\) such that

\[
\liminf_{i \to \infty} a_i \leq \lambda(a) \leq \limsup_{i \to \infty} a_i.
\]

3. Let \(H\) be a Hilbert space over \(\mathbb{R}\) with the property that for any sequence \(\{x_n\}_{n=1}^\infty \subset H\) with \(\|x_n\| \leq 1\), there exists a convergent subsequence. Prove that \(H\) is finite dimensional.