Spring 2019 Math 215C: Homework 1
(Due 4/18 during lecture)

1. (a) Let $M^n$ and $N^n$ be smooth manifolds with atlases $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in \mathcal{B}}$ respectively. Prove that $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}}$ with $\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^{m+n}$ given by

$$(\phi_\alpha \times \psi_\beta)(x, y) := (\phi_\alpha(x), \psi_\beta(y)),$$

is an atlas on $M \times N$.

(b) Define $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{Z}^n$ acts properly discontinuously on $\mathbb{R}^n$ by

$$\varphi_{(m_1, \ldots, m_n)}(x^1, \ldots, x^n) := (x^1 + m^1, \ldots, x^n + m^n).$$

Prove that $\mathbb{T}^n$ is diffeomorphic to $S^1 \times \cdots \times S^1$ (n times).

2. Let $M^n$ be a smooth manifold, and $TM$ be the tangent bundle. Suppose $\{(U_\alpha, \varphi_\alpha)\}$ is an atlas on $M$. Define an atlas $\{((U_\alpha, \tilde{\varphi}_\alpha)) \} \in TM$ with $U_\alpha := \cup_{p \in U_\alpha} T_p M$ and $\tilde{\varphi}_\alpha : U_\alpha \to \mathbb{R}^{2n}$ given by $\tilde{\varphi}_\alpha(c, \frac{\partial}{\partial x^i}) := (\varphi_\alpha(p), c^1, \ldots, c^n)$. [You may use without proof that this is indeed an atlas.]

(a) Compute the transition maps $\tilde{\varphi}_\alpha \circ \tilde{\varphi}^{-1}_\beta$.

(b) Show that $TM$ is orientable (regardless of whether $M$ is orientable).

3. (For this problem, you may use the inverse function theorem and/or implicit function theorem in $\mathbb{R}^n$ without proof.) Let $M^n$, $N^n$ be smooth manifolds. Suppose $\varphi : N \to M$ is an immersion (so that $n \leq m$).

(a) Let $q \in N$. Prove that there exists an open set $V \subset N$ containing $q$ such that $\varphi | V$ is an embedding.

(b) Given $V$ as above, let $p \in \varphi(V)$. Prove that there is an open set $U \subset M$ containing $p$ and a smooth map $\psi : U \to \mathbb{R}^m$ such that $U$ is diffeomorphic to $\psi(U)$ and $\psi(\varphi(V) \cap U) = \{(x^1, \ldots, x^m) : x^{n+1} = \cdots = x^n = 0\}$.

(c) Suppose in addition that $g$ is a Riemannian metric on $M$. Prove that for $U$ as in part (b), there exist $m$ smooth vector fields $E_1, E_2, \ldots, E_m$ on $U$ such that

i. at any point on $\varphi(V) \cap U$, $E_1, E_2, \ldots, E_n$ are tangent to $\varphi(N)$, and

ii. $g(E_i, E_i) = 1$ and $g(E_i, E_j) = 0$ if $i \neq j$. [Hint: Gram–Schmidt!]

4. (a) Define the Minkowski spacetime to be the Lorentzian manifold $\mathbb{R}^2$ with metric $g = -dt^2 + dx^2$. Prove that for any $v \in (-1, 1)$, the map from the Minkowski spacetime to itself given by $t \mapsto t' := \frac{t}{\sqrt{1-v^2}}(t-vx), x \mapsto x' := \frac{1}{\sqrt{1-v^2}}(x-vt)$ is an isometry.
(b) Equip $\mathbb{S}^n := \{(x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} (x_i)^2 = 1\}$ with the round metric. Let $A$ be an $(n \times n)$ matrix such that $A^T A = I$. Define $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $F(x) = Ax$ and define $G : \mathbb{S}^n \to \mathbb{S}^n$ as the restriction of $F$ on $\mathbb{S}^n$. Show that $G$ is an isometry. [In fact all isometries of $\mathbb{S}^n$ are of this form. You can already try to prove this. We will return to this again when we have more tools and this question becomes easier.]

5. Let $(M, g)$ be a Riemannian manifold and $H$ be a group. Suppose $H$ acts on $M$ properly discontinuously and such that the action $\phi_h : M \to M$ is an isometry for all $h \in H$. Prove that there exists a unique Riemannian metric $r$ on $M/H$ such that the projection map $\pi : M \to M/H$ satisfies $g(X, Y) = r((d\pi)_p(X), (d\pi)_p(Y))$ for every $X, Y \in T_p M$ and $p \in M$. [You can use the fact that $\pi : M \to M/H$ is a smooth map without proof. Note that this for instance gives a metric on $\mathbb{R}P^n$.]

6. Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ as in Problem 1. Using the standard metric $g_{\mathbb{R}^n}$ on $\mathbb{R}^n$ and problem 5, we define a Riemannian metric $g_{T^n}$ on $T^n$.

Consider the map $F : (T^n, g_{T^n}) \to (\mathbb{R}^{2n}, g_{\mathbb{R}^{2n}})$ defined by

$$F(x^1, \ldots, x^n) = \left(\frac{\cos(2\pi x^1)}{2\pi}, \frac{\sin(2\pi x^1)}{2\pi}, \ldots, \frac{\cos(2\pi x^n)}{2\pi}, \frac{\sin(2\pi x^n)}{2\pi}\right).$$

Prove that $F$ is an isometric embedding (i.e. $F : T^n \to \mathbb{R}^{2n}$ is an embedding and $F : T^n \to \text{im}(T^n)$ is an isometry). [Note that $F$ is well-defined. You need not show this fact explicitly.]