Spring 2019 Math 215C: Homework 4
(Due 5/30 during lecture)

1. (a) (You do not need to turn this part in.) Let \((\mathcal{M}, \mathcal{g})\) be a Lorentzian manifold and \((M, g)\) be an embedded Riemannian submanifold such that \(g\) is the induced metric (i.e. denoting the embedding \(\iota : M \to \mathcal{M}, g(X, Y) = \mathcal{g}(\iota_*(X), \iota_*(Y))\). Check that the Gauss equation holds in this setting (with the same proof).

(b) Let \(\mathcal{M} = \mathbb{R}^{n+1}\) with the metric \(\mathcal{g} = (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2\). Define \(M = \{(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : (x^{n+1})^2 - \sum_{i=1}^n(x^i)^2 = 1\}\) and suppose \(g\) is the induced Riemannian metric. Show that \((M, g)\) has constant sectional curvature \(-1\). [This gives yet another way of representing the hyperbolic space.]

2. Let \((M, g)\) be a Riemannian manifold, \(\gamma : [0, 1] \to M\) be a geodesic, and \(J\) be a Jacobi field along \(\gamma\). Prove that there exists a parametrized surface \(f(s,t)\), where \(f(0,t) = \gamma(t)\) and the curves \(t \mapsto f(s,t)\) are geodesics such that \(J(t) = \frac{\partial f}{\partial s}(0,t)\). [Hint: Choose a curve \(\lambda(s), s \in (-\epsilon, \epsilon)\) in \(M\) such that \(\lambda(0) = \gamma(0)\) and \(\lambda'(0) = J(0)\). Along \(\lambda\) choose a vector field \(W(s)\) with \(W(0) = \gamma'(0), D_s W(0) = D_s J(0)\). Define \(f(s,t) = \exp_{\gamma(t)} t W(s)\) and verify that \(\partial_s f(0,0) = \partial_s \lambda(0) = J(0)\) and \(D_s \partial_s f(0,0) = D_s \partial_s f(0,0) = D_s W(0) = D_s J(0)\).]

3. Given a Riemannian manifold \((M, g)\) show that, in normal coordinates centered at \(p \in M\), the metric \(g\) admits the expansion

\[
g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{kilj}(p)x^k x^l + \mathcal{O}(|x|^3),
\]

around \(p = (0, \ldots, 0)\). [Hint: fix \((x^1, \ldots, x^n)\) and show that the vectors \(J_1(t) := t \partial x^1, \ldots, J_n(t) := t \partial x^n\) are Jacobi fields along the geodesic \(t \mapsto (tx^1, \ldots, tx^n)\). Then Taylor expand the functions \(f_{ij}(t) := g(J_i(t), J_j(t))\) around \(t = 0\).]

4. Given a Riemannian manifold \((M, g)\) and a 2-plane \(\Pi_p \subset T_p M\), let \(C^0_r \subset \Pi_p\) denote the circle of radius \(r\) centered at the origin, and let \(C_r := \exp_{p}(C^0_r)\) denote the image of \(C^0_r\) under the exponential map at \(p\). Show that

\[
\lim_{r \to 0} \frac{2 \pi r - L(C_r)}{r^3} = \frac{\pi}{3} \kappa(\Pi_p),
\]

where \(L(C_r)\) denotes the length of \(C_r\) and \(\kappa(\Pi_p)\) denotes the sectional curvature of \(\Pi_p\). In other words, sectional curvature has the geometric interpretation of infinitesimally measuring the deviation of the length of the geodesic circle \(C_r\) with the circle of radius \(r\) in Euclidean space. [Hint: let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of \(T_p M\) such that \(\Pi_p = \text{span}\{e_1, e_2\}\), so that, in normal coordinates defined by \(\{e_1, \ldots, e_n\}\) centered at \(p\), \(C_r = \{(r \cos \theta, r \sin \theta, 0, \ldots, 0) \mid \theta \in [0, 2\pi]\}\). Then use the previous problem.]

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