MATH 220: FINAL EXAM – DECEMBER 11, 2009 – SOLUTIONS

This is a closed book, closed notes, no calculators exam.
There are 7 problems. Solve all of them. Total score: 200 points.

Problem 1.  
(i) (20 points) For \(|y - 1|\) small, solve 
\[ xu_x + yu_y = 1, \quad u(x, 1) = x^2. \]

Sketch the characteristics, and discuss where in \(\mathbb{R}^2\) is the solution uniquely determined by the initial data. Does the solution you found extend to this region? Does it extend to a larger region?

(ii) (15 points) For \(|x|\) small, solve 
\[ uu_x + uu_y = 1, \quad u(0, y) = y^2 + 1. \]

Solution 1.  
(i) This is a linear first order PDE. The characteristic ODEs are 
\[ \frac{dx}{ds} = x, \quad \frac{dy}{ds} = y, \quad \frac{dz}{ds} = 1, \]
with initial condition 
\[ x(r, 0) = r, \quad y(r, 0) = 1, \quad z(r, 0) = r^2. \]

The ODEs are easily solved to yield 
\[ x(r, s) = re^s, \quad y(r, s) = e^s, \quad z(r, s) = s + r^2. \]

Inverting the map \((r, s) \mapsto (x, y)\) yields \(r = x/y, \quad s = \log y\). Thus, 
\[ u(x, y) = z(r(x, y), s(x, y)) = \log y + (x/y)^2; \]

this indeed solves the PDE in \(y > 0\). Now note that where \(y \neq 0\) the characteristics are given by constant \(x/y\) (with \(y\) having a fixed sign), and where \(x \neq 0\) they are given by constant \(y/x\) (with \(x\) having a fixed sign), thus they are half lines emanating from the origin. Since for every \((x_0, y_0)\) with \(y_0 > 0\) these half line through \((x_0, y_0)\) also crosses the line \(y = 1\), the solution is uniquely determined in \(y > 0\), but not elsewhere for the other characteristics do not cross the initial curve. Nonetheless, along the \(x\) axis, \(y = 0\), the solutions cannot be continuous, for the solution determined in \(y > 0\) does not stay bounded as \(y \to 0\) as long as \(x \neq 0\).

(ii) This is a quasilinear first order PDE. The characteristic ODEs and initial conditions are 
\[ \frac{dx}{ds} = z, \quad x(r, 0) = 0, \]
\[ \frac{dy}{ds} = z, \quad y(r, 0) = r, \]
\[ \frac{dz}{ds} = 1, \quad z(r, 0) = r^2 + 1. \]

The \(z\) ODE yields 
\[ z(r, s) = r^2 + 1 + s. \]

The \(x\) and \(y\) ODEs then become 
\[ \frac{dx}{ds} = r^2 + 1 + s, \quad x(r, 0) = 0 \Rightarrow x(r, s) = s(r^2 + 1) + \frac{s^2}{2}, \]
\[ \frac{dy}{ds} = r^2 + 1 + s, \quad y(r, 0) = r \Rightarrow y(r, s) = s(r^2 + 1) + \frac{s^2}{2} + r. \]
We now need to express \( r, s \) in terms of \( x, y \). First,

\[ r = y - x, \]

so the solution of the ODE for \( x \) yields

\[ \frac{s^2}{2} + s((y - x)^2 + 1) - x = 0, \]

and thus

\[ s = -((y - x)^2 + 1) \pm \sqrt{((y - x)^2 + 1)^2 + 2x}. \]

To see which sign we actually want, note that when \( x = 0 \) we need \( s = 0 \), so \( 0 = -(y^2 + 1) \pm \sqrt{(y^2 + 1)^2} \) which implies that (with \( \sqrt{\cdot} \) denoting the non-negative square root) we need the + sign. Therefore

\[ u(x, y) = (y - x)^2 + 1 - ((y - x)^2 + 1) + \sqrt{((y - x)^2 + 1)^2 + 2x} = \sqrt{((y - x)^2 + 1)^2 + 2x}. \]

A different way of reaching the same solution is to note that the PDE says

\[ \frac{1}{2} ((u^2)_x + (u^2)_y) = 1, \]

so if we let \( v = u^2 \), we have the linear PDE

\[ v_x + v_y = 1, \quad v(0, y) = (y^2 + 1)^2. \]

The method of characteristics now gives

\[
\begin{align*}
\frac{\partial x}{\partial s} &= 1, \quad x(r, 0) = 0, \\
\frac{\partial y}{\partial s} &= 1, \quad y(r, 0) = r, \\
\frac{\partial z}{\partial s} &= 2, \quad z(r, 0) = (r^2 + 1)^2.
\end{align*}
\]

Thus, \( x = s, \ y = s + r, \) so \( s = x, \ r = y - x, \) and then \( z = 2s + (r^2 + 1)^2, \) so

\[ v(x, y) = 2x + ((y - x)^2 + 1)^2. \]

Finally, \( u \) is the positive square root of \( v \), giving the same answer as above.

**Problem 2.** Consider the wave equation \( u_{tt} = c^2 u_{xx} \) on the half-line, i.e. on \([0, \infty) \times [0, \infty)\), with homogeneous Dirichlet boundary condition \( u(0, t) = 0 \), and with initial conditions \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \) for \( x \geq 0 \).

(i) (10 points) Find \( u \).

(ii) (8 points) Suppose \( \phi, \psi \) are both linear near 0 (i.e. \( \phi(x) = cx \) for \( x \) small, and similarly for \( \psi \)), and are \( C^\infty \) away from a point \( x_0 > 0 \). Where can you say for sure that \( u \) is \( C^\infty \)?

(iii) (7 points) Suppose that \( \phi \equiv 0 \), and \( \psi(x) = x \) for \( x < 1 \), \( \psi(x) = 0 \) for \( x > 1 \). Find \( u(x, t) \) explicitly for \( t \geq 0 \). (Hint: it is best to consider different cases depending on where \( (x, t) \) lies.) Does the location of the singularities (lack of being \( C^\infty \)) agree with what you found in (ii)?

You may use in any part of the problem that if \( v \) solves \( v_{tt} - c^2 v_{xx} = 0 \) on \( \mathbb{R}^2 \) then

\[ v(x, t) = \frac{v(x - ct, 0) + v(x + ct, 0)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} v_t(x', 0) \, dx'. \]

**Solution 2.**

(i) Let \( v \) be the odd extension of \( u \), i.e. \( v(x, t) = u(x, t) \) for \( x \geq 0 \), \( v(x, t) = -u(-x, t) \) for \( x < 0 \), and let \( \phi_o, \psi_o \) be the odd extensions of \( \phi, \psi \) to \( \mathbb{R} \). Then \( v \) solves the wave equation with initial data \( \phi_o, \psi_o \), so

\[ v(x, t) = \frac{\phi_o(x - ct) + \phi_o(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_o(y) \, dy. \]
Then for $x \geq 0$, $u(x, t) = v(x, t)$, so for $x > ct$,

$$u(x, t) = \frac{\phi(x - ct) + \phi(x + ct)}{2} + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(y) \, dy,$$

while for $0 \leq x < ct$, $\phi_o(x - ct) = -\phi(ct - x)$, and

$$\int_{x - ct}^{x + ct} \psi_o(y) \, dy = -\int_{0}^{ct-x} \psi(-y) \, dy + \int_{0}^{x + ct} \psi(y) \, dy,$$

$$= -\int_{0}^{ct-x} \psi(y) \, dy + \int_{0}^{x + ct} \psi(y) \, dy = \int_{ct-x}^{x + ct} \psi(y) \, dy,$$

so

$$u(x, t) = \frac{-\phi(ct - x) + \phi(x + ct)}{2} + \frac{1}{2c} \int_{ct-x}^{x + ct} \psi(y) \, dy.$$

(ii) Under these assumptions, $\phi_o$ and $\psi_o$ are $C^\infty$ except at $\pm x_0$, so $v(x, t)$ is $C^\infty$ except on the characteristics through these two points, i.e. $x \pm ct = x_0$ and $x \pm ct = -x_0$, hence $u$, being the restriction of $v$, is $C^\infty$ except at where these characteristics intersect $x \geq 0$, i.e. along the broken characteristics emanating from $x_0$.

(iii) It is easier to find $v$ in $x > 0$ directly rather than apply the formula we derived above. Thus, $\psi_o(x) = x$ if $|x| < 1$, 0 otherwise, so the answer depends on where $x \pm ct$ are relative to the interval $(-1, 1)$. If $x \geq 0$, we have the following cases:

$$u(x, t) = \begin{cases} 
0, & 0 < x - ct, \\
\frac{1}{2c}(1 - (x - ct)^2), & -1 < x - ct < 1 < x + ct, \\
\frac{1}{4c}((x + ct)^2 - (x - ct)^2) = xt, & -1 < x - ct < x + ct < 1, \\
0, & x - ct < -1 < x + ct. 
\end{cases}$$

This is certainly smooth except where two regions meet, which is exactly along the characteristic lines emanating from $\pm 1$.

**Problem 3.** (25 points) Consider the (real-valued) damped wave equation on $[0, \ell]_x \times [0, \infty)_t$ with Robin boundary conditions:

$$u_{tt} + a(x)u_t = (c(x)^2u_x)_x, \quad u_x(0,t) = \alpha u(0,t), \quad u_x(\ell,t) = -\beta u(\ell,t)$$

where $\alpha, \beta \geq 0$ are constants, $a \geq 0$ and $c > 0$ depend on $x$ only, and there are constants $c_1, c_2 > 0$ such that $c_1 \leq c(x) \leq c_2$ for all $x$. (Note that if $\alpha = 0$ and $\beta = 0$ then this is just the Neumann boundary condition! In general, this BC would hold for example for a string if its ends were attached to springs.) Assume throughout that $u$ is $C^2$. Let

$$E(t) = \frac{1}{2} \int_{0}^{\ell} (u_t(x,t)^2 + c(x)^2u_x(x,t)^2) \, dx + \frac{1}{2} (c(0)^2\alpha u(0,t)^2 + c(\ell)^2 \beta u(\ell,t)^2).$$

(i) Show that if $\alpha = 0$ then $E$ is constant.

(ii) Show that if $\alpha \geq 0$ then $E$ is a decreasing (i.e. non-increasing) function of $t$, and that the solution of the damped wave equation (under the conditions mentioned above) with given initial condition is unique.

(Note that, in the physical example, the extra terms in the energy represent the potential energy stored in the springs at the end of the string.)

**Solution 3.** Differentiation gives

$$E'(t) = \int_{0}^{\ell} (u_t u_{tt} + c(x)^2u_x u_{xt}) \, dx + (c(0)^2\alpha u(0,t)u_t + c(\ell)^2 \beta u(\ell,t)u_t).$$
Using the PDE to rewrite $E'(t)$,

$$E'(t) = \int_0^\ell (u_t(c(x)x^2u_x)_x - a(x)u_t^2 + c(x)^2u_x(x,t)u_{xt})\,dx$$
$$+ (c(0)^2\alpha u(0,t)u_t(0,t) + c(\ell)^2\beta u(\ell,t)u_t(\ell,t))$$
$$= \int_0^\ell ((c(x)^2u_xu_t)_x - a(x)u_t^2)\,dx + (c(0)^2\alpha u(0,t)u_t(0,t) + c(\ell)^2\beta u(\ell,t)u_t(\ell,t)).$$

Using the fundamental theorem of calculus to evaluate the integral of the derivative,

$$E'(t) = c(x)^2u_xu_t|_0^\ell - \int_0^\ell a(x)u_t^2\,dx + (c(0)^2\alpha u(0,t)u_t(0,t) + c(\ell)^2\beta u(\ell,t)u_t(\ell,t))$$
$$= -\int_0^\ell a(x)u_t^2\,dx + c(\ell)^2u_x(\ell,t)u_t(\ell,t) - c(0)^2u_x(0,t)u_t(0,t)$$
$$+ (c(0)^2\alpha u(0,t)u_t(0,t) + c(\ell)^2\beta u(\ell,t)u_t(\ell,t)).$$

Using the boundary conditions thus yields

$$E'(t) = -\int_0^\ell a(x)u_t^2\,dx - \beta c(\ell)^2u(\ell,t)u_t(\ell,t) - \alpha c(0)^2u(0,t)u_t(0,t)$$
$$+ (c(0)^2\alpha u(0,t)u_t(0,t) + c(\ell)^2\beta u(\ell,t)u_t(\ell,t))$$
$$= -\int_0^\ell a(x)u_t^2\,dx.$$

If $a \equiv 0$, we deduce that $E'(t) = 0$, i.e. $E$ is independent of $t$. In general, if $a \geq 0$, we deduce that $E'(t) \leq 0$, hence $E(t) \leq E(0)$ for $t \geq 0$. In particular, if $E(0) = 0$ then, taking into account that $E(t)$ is non-negative in view of its definition (as $\alpha, \beta \geq 0$), $0 \leq E(t) \leq 0$, thus $E(t)$ vanishes identically. But this gives that $u_t$ and $u_x$ vanish identically from the definition of $E$, and so $u$ is a constant. So now suppose that $u_1$ and $u_2$ solve the PDE with the same initial condition, and let $u = u_1 - u_2$. Thus, $u(x,0) = 0$ and $u_t(x,0) = 0$, so also $u_x(x,0) = 0$, and thus $E(0) = 0$. We deduce that $u$ is a constant, so as $u$ vanishes when $t = 0$, this constant is 0. Thus, $u_1 \equiv u_2$, showing the claimed uniqueness.

**Problem 4.**

(i) (8 points) Consider the following eigenvalue problem on $[0, \ell]$:

$$-X'' = \lambda X, \quad X(0) = 0, \quad X'(\ell) = 0.$$

Find all eigenvalues and eigenfunctions, and show that eigenfunctions corresponding different eigenvalues are orthogonal to each other.

(ii) (8 points) Using separation of variables, find the general ‘separated’ solution of the heat equation (with $k > 0$ fixed):

$$u_t = ku_{xx}, \quad u(0,t) = 0, \quad u_x(x,t) = 0.$$

(iii) (6 points) Solve the heat equation with initial condition

$$u(x,0) = \phi(x),$$

i.e. give a formula for the series coefficients in part (ii) in terms of $\phi$.

(iv) (8 points) Now suppose $\phi(x) = x(\ell-x)^2$. Give an estimate for the coefficients in the series which implies the uniform convergence of the series on $[0, \ell] \times [0, \infty)$, and explain how the estimate implies uniform convergence. You do not need to compute the coefficients, though that is one way of getting the desired estimate.

**Solution 4.**

(i) As both the Dirichlet and Neumann boundary conditions are symmetric, the operator $A$ given by $AX = -X''$ on the domain

$$D = \{X \in C^2([0, \ell]) : X(0) = 0, \quad X'(\ell) = 0\}$$
is symmetric. Thus, all eigenvalues of $A$ are real, and eigenfunctions corresponding to
different eigenvalues are orthogonal to each other. To find the eigenfunctions, note that
the general solution of $-X'' = \lambda X$ is, for $\lambda \neq 0$,

$$X(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x),$$

so $X(0) = 0$ gives $A = 0$, while $X'(\ell) = 0$ gives (assuming $B \neq 0$)

$$\sqrt{\lambda} \ell = \left( n + \frac{1}{2} \right) \pi,$$

with $n$ an integer. As the corresponding eigenfunctions are

$$X_n(x) = \sin \left( \frac{(n + \frac{1}{2}) \pi x}{\ell} \right),$$

allowing $n < 0$ gives the same functions as $n \geq 0$, so we may restrict to $n \geq 0$. For $\lambda = 0$
the general solution is $X(x) = A + Bx$, and the boundary conditions give $A = 0$ and
$B = 0$, so 0 is not an eigenvalue.

(ii) Writing $u(x, t) = X(x)T(t)$, substituting into the PDE yields $XT' = kX''T$, so
$\frac{X'}{X} = \frac{-k}{X''}$, and as the left hand side is independent of $x$, the right hand side is independent
of $t$, they are both constants, $\lambda$. We must also have, from the homogeneous boundary
conditions, $X(0) = 0 = X'(\ell)$, so $X$ is one of the eigenfunctions computed in the previous
part, with eigenvalue

$$\lambda = \lambda_n = \left( \frac{(n + \frac{1}{2}) \pi}{\ell} \right)^2.$$

On the other hand, the $T$ ODE gives then

$$T(t) = Ae^{-\lambda t},$$

so the general separated solution is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \exp \left( - \left( \frac{(n + \frac{1}{2}) \pi}{\ell} \right)^2 k t \right) \sin \left( \frac{(n + \frac{1}{2}) \pi x}{\ell} \right).$$

(iii) Letting $t = 0$ gives

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin \left( \frac{(n + \frac{1}{2}) \pi x}{\ell} \right).$$

As the sines are orthogonal to each other, as remarked in the first part, and as

$$\int_0^{\ell} \sin \left( \frac{(n + \frac{1}{2}) \pi x}{\ell} \right)^2 dx = \frac{\ell}{2},$$

we deduce that

$$A_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin \left( \frac{(n + \frac{1}{2}) \pi x}{\ell} \right) dx.$$

(iv) Writing $X_n$, $\lambda_n$ as above, we have

$$A_n = \frac{\langle \phi, X_n \rangle}{\|X_n\|^2} = \frac{2}{\ell} \langle \phi, X_n \rangle.$$

Thus, using the symmetry of the operator $A$ and the fact that $\phi \in D$,

$$A_n = \frac{2}{\ell \lambda_n} \langle \phi, \lambda_n X_n \rangle = \frac{2}{\ell \lambda_n} \langle \phi, AX_n \rangle = \frac{2}{\ell \lambda_n} \langle \phi, AX_n \rangle = \frac{2}{\ell \lambda_n} \langle A \phi, X_n \rangle,$$

so

$$|A_n| \leq \frac{2}{\ell \lambda_n} \int_0^{\ell} |\phi''(x)||X_n(x)| dx \leq \frac{2\ell^2}{(n + \frac{1}{2})^2 \pi^2} \sup |\phi''(x)|.$$
Thus, $|A_n| \leq C/(n+1/2)^2$, and therefore the terms in the series for $u(x,t)$ satisfy
$$
|A_n \exp \left( - \left( \frac{n+\frac{1}{2}}{\ell} \right)^2 kt \right) \sin \left( \frac{(n+\frac{1}{2})\pi x}{\ell} \right) | \leq C/(n+1/2)^2 = M_n.
$$
Since $\sum M_n$ converges, by the Weierstrass M-test the series indeed converges uniformly.

**Problem 5.**
(i) (15 points) For both of the following functions $f$ on $[0,\ell]$, state whether the Fourier sine series on $[0,\ell]$ converges in each of the following senses: uniformly, in $L^2$. State what the Fourier series converges to on all of $\mathbb{R}$. Make sure that you give the reasoning that led you to the conclusions.

(a) $f(x) = x^2(\ell - x)^4$,
(b) $f(x) = 0$, for $0 \leq x \leq \ell/2$, and $f(x) = x - \ell/2$ for $\ell/2 < x \leq \ell$.

(ii) (10 points) For the function $f$ in (b) above, we wish to approximate $f$ by a function $g$ of the form $a_1 \sin(\pi x/\ell) + a_3 \sin(3\pi x/\ell)$ on $[0,\ell]$. Find the constants $a_1$ and $a_3$ that minimize the $L^2$ error, $\int_0^\ell |f - g|^2 \, dx$, of the approximation.

**Solution 5.**
(i) The function $f$ in (a) is $C^2$ on $[0,\ell]$ and satisfies Dirichlet boundary conditions (as the sines in the sine series do), so the Fourier sine series converges uniformly to $f$, and hence also in $L^2$. The odd $2\ell$-periodic extension of the function $f$ in (b) is discontinuous: it is odd about $\ell$, and the limit from below is $\ell/2 \neq 0$, so the limit from above is $-\ell/2$. Thus, the Fourier sine series does not converge uniformly (the uniform limit of continuous functions in continuous), but $f$ is piecewise $C^1$, so the Fourier series converges in $L^2$; in either case the uniform, resp. $L^2$, limit of the series on $\mathbb{R}$ is the odd $2\ell$-periodic extension of $f$.

(ii) The choice of $a_1$ and $a_3$ minimizing $\int_0^\ell |f - g|^2 \, dx$ are the ones given by the orthogonal projection of $f$ to the span of $\cos(\pi x/\ell)$ and $\cos(3\pi x/\ell)$, which in turn are simply the Fourier sine coefficients. These are
$$
\frac{2}{\ell} \int_0^\ell f(x) \sin(n\pi x/\ell) \, dx = \frac{2}{\ell} \int_{\ell/2}^\ell (x - \ell/2) \sin(n\pi x/\ell) \, dx
$$
$$
= -\frac{2}{n\pi} (x - \ell/2) \cos(n\pi x/\ell)|_{x=\ell/2} + \frac{2}{(n\pi)^2} \sin(n\pi x/\ell)|_{x=\ell/2}.
$$
But $n = 1, 3$, so
$$
a_1 = \left( \frac{1}{\pi} - \frac{2}{\pi^2} \right) \ell, \quad a_3 = \left( \frac{1}{3\pi} + \frac{2}{9\pi^2} \right) \ell.
$$

**Problem 6.** Recall that $\mathcal{S}(\mathbb{R}^n)$ is the set of Schwartz functions on $\mathbb{R}^n$.

(i) (7 points) Show that if $\phi, \psi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x), (1 + |x|)^N \psi(x)$ both bounded for some $N > n$ then
$$
\int_{\mathbb{R}^n} (\mathcal{F}\phi)(\xi) \psi(\xi) \, d\xi = \int_{\mathbb{R}^n} \phi(x) (\mathcal{F}\psi)(x) \, dx.
$$

(ii) (6 points) Define $\mathcal{F}u$ if $u$ is a tempered distribution, i.e. $u \in \mathcal{S}'(\mathbb{R}^n)$, and show that this is consistent with the standard definition if $u = \iota_\phi$, $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$.

(iii) (5 points) Recall that $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ means that for each $\phi \in \mathcal{S}(\mathbb{R}^n)$, $u_j(\phi) \to u(\phi)$.

(iv) (5 points) Show that for $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$,
$$
\frac{1}{(2\pi)^n} \phi(\xi) = (2\pi)^n (\mathcal{F}^{-1}\phi)(\xi).
$$

(v) (7 points) Show the Parseval/Plancherel formula, i.e. that for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$,
$$
\int_{\mathbb{R}^n} \phi(x) \overline{\psi(x)} \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathcal{F}\phi)(\xi) (\overline{\mathcal{F}\psi})(\xi) \, d\xi,
$$
and hence conclude that, up to a constant factor, the Fourier transform preserves $L^2$-norms:
\[ \|F\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|\phi\|_{L^2(\mathbb{R}^n)}. \]

**Solution 6.**

(i) Writing the Fourier transform as an integral and using that one can interchange the integrals since the double integral is absolutely convergent as
\[ |\phi(x)\psi(\xi)| \leq C^2(1 + |x|)^{-2N}, \quad 2N > 2n, \]
we deduce that
\[
\int_{\mathbb{R}^n} (F\phi)(\xi) \psi(\xi) \, d\xi = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) \, dx \right) \psi(\xi) \, d\xi = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x)\psi(\xi) \, dx \, d\xi
\]
\[
= \int_{\mathbb{R}^n} \phi(x) \left( \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \, d\xi \right) \psi(\xi) \, dx = \int_{\mathbb{R}^n} \phi(x) (F\psi)(x) \, dx.
\]

(ii) For $u \in S'(\mathbb{R}^n)$, one defines
\[
(Fu)(\psi) = u(F\psi), \quad \psi \in S(\mathbb{R}^n).
\]
Note that this makes sense since $F\psi \in S(\mathbb{R}^n)$. Moreover, if $u = \iota_\phi$, $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$, then, by part (i), for any $\psi \in S(\mathbb{R}^n)$,
\[
\iota_{F\phi}(\psi) = \int_{\mathbb{R}^n} (F\phi)(\xi) \psi(\xi) \, d\xi = \int_{\mathbb{R}^n} \phi(x)(F\psi)(x) \, dx = \iota_\phi(F\psi) = (F\iota_\phi)(\psi),
\]
and hence $F\iota_\phi = \iota_{F\phi}$, i.e. the definition for tempered distributions is consistent with the definition for functions $\phi$ as above.

(iii) If $u_j \to u$ in $S'(\mathbb{R}^n)$ then for $\psi \in S(\mathbb{R}^n)$,
\[
Fu_j(\psi) = u_j(F\psi) \to u(F\psi) = Fu(\psi),
\]
which says exactly that $Fu_j \to Fu$ in $S'(\mathbb{R}^n)$.

(iv) For $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$,
\[
\overline{F\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) \, dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) \, dx = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \overline{\phi}(x) \, dx = (2\pi)^n (F^{-1}\overline{\phi})(\xi).
\]

(v) For $\phi, \psi \in S(\mathbb{R}^n)$,
\[
\int_{\mathbb{R}^n} \phi(x) \overline{\psi(x)} \, dx = \int_{\mathbb{R}^n} \phi(x) (F(F^{-1}\overline{\psi}))(x) \, dx = \int_{\mathbb{R}^n} F\phi(\xi) \overline{(F^{-1}\overline{\psi})(\xi)} \, d\xi
\]
\[
= \int_{\mathbb{R}^n} F\phi(\xi) \overline{(2\pi)^{-n}(F\psi)(\xi)} \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} (F\phi)(\xi) \overline{(F\psi)(\xi)} \, d\xi,
\]
where the first equality follows from $FF^{-1} = \text{Id}$ on $S(\mathbb{R}^n)$, the second from part (i) and the third from (iv). Substituting in $\psi = \phi$ yields that
\[
\|F\phi\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|\phi\|^2_{L^2(\mathbb{R}^n)},
\]
giving the claimed conclusion.

**Problem 7.** In $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_t$, we write points as $(x_1, \ldots, x_n, t)$, and also write $x = (x_1, \ldots, x_n)$. With $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, consider the modified wave equation in $\mathbb{R}^{n+1}$:
\[
u_{tt} - c^2 \Delta_x u - \lambda u = f.
\]

(i) (12 points) Show that if $f \in S(\mathbb{R}^{n+1})$, then (1) has a unique solution $u$ in $S(\mathbb{R}^{n+1})$ when $\text{Im} \lambda \neq 0$, and give an expression for $u$ in terms of $f$. Your final formula may involve the (inverse) Fourier transform. (Hint: use the Fourier transform in all variables!)

(ii) (12 points) Still assuming $\text{Im} \lambda \neq 0$, show that if $\phi, \psi \in S(\mathbb{R}^n)$, $f \equiv 0$ then the PDE (1) together with the initial conditions
\[
\begin{align*}
\begin{array}{ll}
u(0) = \phi(x), & \nu_t(0) = \psi(x),
\end{array}
\end{align*}
\]

Solution 7.

(i) Taking the Fourier transform of both sides, and denoting the variables corresponding to (i.e. dual to) \( t \) by \( \tau \), those dual to \( x_j \) by \( \xi_j \), and writing \( \xi = (\xi_1, \ldots, \xi_n) \), we obtain

\[
( -\tau^2 + c^2|\xi|^2 - \lambda ) \mathcal{F} u = \mathcal{F} f.
\]

Now, as \( \text{Im} \lambda \neq 0 \), the factor in front of \( \mathcal{F} u \) never vanishes, so we can divide by it:

\[
\mathcal{F} u = -\frac{\mathcal{F} f}{\tau^2 - c^2|\xi|^2 + \lambda}.
\]

Moreover, \( |\tau^2 - c^2|\xi|^2 + \lambda| \geq |\text{Im}(\tau^2 - c^2|\xi|^2 + \lambda)| = |\text{Im}\lambda| \), so

\[
|\mathcal{F}u(\xi, \tau)| \leq \frac{|\mathcal{F}f(\xi, \tau)|}{|\text{Im}\lambda|},
\]

so it is rapidly decreasing as \( \mathcal{F} f \) is Schwartz. In particular, even if \( u \) is merely a tempered distribution, it is uniquely determined by \( f \) by this formula. To see that the \( \mathcal{F} u \) is actually Schwartz, we also need to estimate derivatives of \( \mathcal{F} u \); recall that functions which are polynomially bounded with all derivatives map \( \mathcal{S}(\mathbb{R}^n) \) to itself as multiplication operators (this is just the product rule), so one merely needs to check that \( \partial_\xi^k \partial_t^l \left( \tau^2 - c^2|\xi|^2 + \lambda \right)^{-1} \) is polynomially bounded. But this is straightforward: iterated application of the product rule and the quotient rule shows that this derivative is a polynomial divided by a power of \( \tau^2 - c^2|\xi|^2 + \lambda \); since the latter is \( \geq |\text{Im}\lambda| \) in absolute value, the claim follows. Thus, \( \mathcal{F} u \in \mathcal{S}(\mathbb{R}^n) \) and hence \( u \in \mathcal{S}(\mathbb{R}^n) \),

\[
\mathcal{F}^{-1} \left( -\frac{\mathcal{F} f}{\tau^2 - c^2|\xi|^2 + \lambda} \right).
\]

(ii) Now consider the PDE with the initial conditions

\[
u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x).
\]

Take the partial Fourier transform of the PDE in \( x \) only; denote this by \( \hat{u}(\xi, t) \). Thus,

\[
\hat{u}_{tt} + c^2|\xi|^2 \hat{u} - \lambda \hat{u} = 0, \quad \hat{u}(\xi, 0) = \hat{\phi}(\xi), \quad \hat{u}_t(\xi, 0) = \hat{\psi}(\xi).
\]

The general solution of the ODE is (note that as \( \lambda \) is not real, \( c^2|\xi|^2 - \lambda \neq 0 \))

\[
\hat{u}(\xi, t) = A(\xi) \cos(\sqrt{c^2|\xi|^2 - \lambda} t) + B(\xi) \sin(\sqrt{c^2|\xi|^2 - \lambda} t),
\]

where we had to make some (any) choice of the square root, e.g. the square root of \( re^{i\theta} \) is \( \sqrt{r}e^{i\theta/2} \) if \( r > 0 \) and \( \theta \in (-\pi, \pi) \), where the square root is the positive square root for positive numbers. Note that the expression under the square root is never real! The initial conditions give

\[
A(\xi) = \hat{\phi}(\xi), \quad B(\xi) \sqrt{c^2|\xi|^2 - \lambda} = \hat{\psi}(\xi).
\]

Thus,

\[
\mathcal{F}u(\xi, t) = \hat{\phi}(\xi) \cos(\sqrt{c^2|\xi|^2 - \lambda} t) + \hat{\psi}(\xi) \frac{\sin(\sqrt{c^2|\xi|^2 - \lambda} t)}{\sqrt{c^2|\xi|^2 - \lambda}},
\]

even if \( u \) is just a tempered distribution in \( x \), depending smoothly on \( t \). Thus,

\[
u = \phi *_x \mathcal{F}^{-1}_\xi(\cos(\sqrt{c^2|\xi|^2 - \lambda} t)) + \psi *_x \mathcal{F}^{-1}_\xi \left( \frac{\sin(\sqrt{c^2|\xi|^2 - \lambda} t)}{\sqrt{c^2|\xi|^2 - \lambda}} \right).
\]
(iii) The solution in (ii) is not a tempered distribution in $\mathbb{R}^{n+1}$ unless $A$ and $B$ both vanish. To see this, rewrite (2) in terms of complex exponentials:

$$A(\xi) \cos(\sqrt{c^2|\xi|^2 - \lambda} t) + B(\xi) \sin(\sqrt{c^2|\xi|^2 - \lambda} t) = A(\xi) - iB(\xi) \frac{e^{i\sqrt{c^2|\xi|^2 - \lambda} t}}{2} + \frac{A(\xi) + iB(\xi)}{2} e^{-i\sqrt{c^2|\xi|^2 - \lambda} t},$$

and note that the exponents are not pure imaginary. Thus, if $\text{Im} \lambda > 0$, then $\text{Im} \sqrt{c^2|\xi|^2 - \lambda} < 0$, hence the absolute value of the first term grows exponentially as $t \to +\infty$, provided $A(\xi) - iB(\xi) \neq 0$, while the second decays exponentially there, while as $t \to -\infty$ we similarly see that the solution grows exponentially unless $A(\xi) + iB(\xi) = 0$. Combining these two facts, the solution grows exponentially unless $A(\xi) = 0 = B(\xi)$, so the only Schwartz solution is the identically zero solution. A similar argument works if $\text{Im} \lambda < 0$. In fact, with a little more care, as some highly oscillatory exponentially growing functions can be tempered, though this is not the case here, this argument even shows that unless $A$ and $B$ vanish identically, the solution is not even tempered!