Autumn 2019 Math 220/CME 303: Homework
Assignment 1 (Due 10/1 during lecture)

In the following, Vasy 1.1, Vasy 2.4, etc. refers to the problems in the posted
notes on the course website.

1. Vasy 1.1
2. Vasy 2.4
3. Vasy 3.2
4. Vasy 3.4
5. Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth function. Assume moreover that there
   is a $C > 0$ such that $|u(x)| \leq C$ for every $x \in \mathbb{R}^n$. Consider the PDE
   \[ \partial_t \phi + u(x) \cdot \nabla \phi = 0, \]
   with initial conditions $\phi(0, x) = f(x)$ smooth, where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.
   (a) (Optional) Prove that the solution exists for all of $\mathbb{R} \times \mathbb{R}^n$.
   (b) Prove the following maximum principle for every $t \in \mathbb{R}$:
       \[ \sup_x |\phi(t, x)| \leq \sup_x |f(x)|. \]
   (c) Prove that the equation preserves positivity in the following sense:
       \[ f(x) > 0 \text{ for all } x \in \mathbb{R}^n \implies \phi(t, x) > 0 \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^n. \]
6. In this problem, $h, r, f, g : \mathbb{R} \rightarrow \mathbb{R}$ are $C^\infty$ and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is $C^\infty$.
   (a) Solve
       \[ \partial_t \eta - \partial_x \eta = 0, \quad \eta(0, x) = h(x). \]
   (b) Solve
       \[ \partial_t \xi + \partial_x \xi = F(t, x), \quad \xi(0, x) = r(x). \]
   (c) Solve the wave equation $\partial^2_{tt} \phi - \partial^2_{xx} \phi = 0$ subject to the initial condi-
       tions $\phi(0, x) = f(x)$ and $(\partial_t \phi + \partial_x \phi)(0, x) = g(x)$. [Hint: Write
       \[ \partial^2_{tt} \phi - \partial^2_{xx} \phi = (\partial_t \partial_t - \partial_x \partial_x) \phi, \]
       use part (a), and then use part (b).]