1. Solve the PDE \( u_{xx} - 2 u_{xy} - 3 u_{yy} = 0 \) with the initial conditions \( u(r, 2r) = 0 \) and \( u_x(r, 2r) = 2r^2 \).

2. (a) Given \( u \in D'(\mathbb{R}^n) \). Define the distribution \( \partial_i u \). [You only need to write down the definition without proving that it is a distribution.]

(b) Let \( u \in D'(\mathbb{R}^2) \) be defined by

\[ u(\varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} \varphi(x, y) \, dy \, dx \]

for every \( \varphi \in C^\infty_c(\mathbb{R}^2) \). Compute \( \partial_x u \) and simplify your answer.

3. Let \( f \in C(\mathbb{R}^n) \) (i.e. \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous function). Suppose that for every \( \varphi \in C^\infty_c(\mathbb{R}^n) \)

\[ \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx = 0. \]

Prove that \( f(x) = 0 \) for every \( x \). [You may use without proof the fact that for every \( x_0 \in \mathbb{R}^n \) and \( \epsilon > 0 \), there is a function \( \chi \in C^\infty_c(\mathbb{R}^n) \) with the following properties: (1) \( \chi(x) \geq 0 \) for all \( x \), (2) \( \int_{\mathbb{R}^n} \chi(x) \, dx = 1 \), (3) \( \text{supp}(\chi) \subset B(x_0, \epsilon) \).]

4. Consider the initial value problem \( u_t + uu_x = 0 \) and \( u(0, x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \in [0, 1] \\ 0 & \text{if } x > 1 \end{cases} \). Consider the piecewise continuous function \( u \) defined by

\[ u(t, x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{t}{2} & \text{if } 0 < x < t \\ 1 & \text{if } t < x < 1 + \frac{t}{2} \\ 0 & \text{if } x > 1 + \frac{t}{2} \end{cases} \text{ for } 0 \leq t < 2, \]

\[ u(t, x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < \sqrt{2t} \\ 0 & \text{if } x > \sqrt{2t} \end{cases} \text{ for } t \geq 2. \]

Prove that \( u \) is a weak solution to the given initial value problem. [You may prove it using a result proved in HW about the Rankine–Hugoniot jump condition, provided that the result is clearly stated.]

5. Consider the boundary value problem for the Poisson equation in a bounded open domain \( \Omega \subseteq \mathbb{R}^2 \) with smooth boundary \( \partial\Omega \):

\[ \begin{cases} \partial^2_{xx} u + \partial^2_{yy} u = f & \text{in } \Omega, \\ u \mid_{\partial\Omega} = 0, \end{cases} \quad (1) \]

where \( f \) is a smooth function.

(a) Prove that there exists a constant \( C_1 > 0 \) such that the following inequality holds for any \( C^1 \) function \( u \) with \( u \mid_{\partial\Omega} = 0. \)

\[ \int_{\Omega} u^2(x, y) \, dx \, dy \leq C_1 \int_{\Omega} [(\partial_x u)^2 + (\partial_y u)^2](x, y) \, dx \, dy. \]

(b) Using part (a), prove that there exists a constant \( C_2 > 0 \) such that the following inequality holds for any \( C^2 \) solution \( u \) solving (1):

\[ \int_{\Omega} [(\partial_x u)^2 + (\partial_y u)^2 + u^2](x, y) \, dx \, dy \leq C_2 \int_{\Omega} f^2(x, y) \, dx \, dy. \]

[You may use the result in part (a) to deduce part (b) even if you did not solve part (a).]