1. Consider the PDE \( u_t + xu_x = \frac{u^3 \sin(e^u)}{1 + u^2} \) with the initial condition \( u(0, x) = \phi(x) \) for smooth \( \phi \). Prove or disprove: for every \( T > 0 \), there is a \( C^1 \) solution to this problem in \((-T, T) \times \mathbb{R}\).

2. Let \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a smooth function such that
\[
|f(x)| \leq \frac{1}{|x|^n}
\]
and (for \( d\sigma \) being the standard measure on the sphere \( \{|x| = r\} \),)
\[
\int_{|x|=r} f(x) \, d\sigma = 0, \quad \forall r \in (0, 1].
\]
Prove that \( u \) defined by
\[
u(\varphi) = \lim_{\epsilon \to 0_+} \int_{|x|\geq \epsilon} f(x) \varphi(x) \, dx
\]
is a distribution.

3. Let \( \{f_n\}_{n=1}^{\infty} \subseteq C(\mathbb{R}^n) \) and \( f \in C(\mathbb{R}^n) \) be functions such that \( f_n(x) \to f(x) \) for all \( x \) and \( f_n(x) \leq f_{n+1}(x) \) for all \( x \) and \( n \). Prove that
\[
\lim_{n \to +\infty} \iota_{f_n}(\varphi) = \iota_f(\varphi)
\]
for every \( \varphi \in C_c(\mathbb{R}^n) \).

4. Find a piecewise continuous function \( u \) which is a weak solution to the problem \( \partial_t u + \frac{1}{2} \partial_x u^2 = 0 \) with initial data
\[
u(0, x) = \begin{cases} 0 & \text{if } x < 0 \\ -\frac{2}{\sqrt{3}} \sqrt{x} & \text{if } x > 0 \end{cases}
\]

5. Consider the initial boundary problem value for the nonlinear equation in a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \):
\[
\begin{aligned}
\partial_t u &= \partial_{xx}^2 u + \partial_{yy}^2 u + u^2 e^{u} \quad \text{in } (0, T) \times \Omega, \\
\partial_t u &= \phi(x, y) \quad \text{in } \Omega, \\
\partial_t u \big|_{\partial \Omega} &= 0,
\end{aligned}
\]
where \( \phi \) is a smooth non-negative function. Suppose (1) admits a solution \( u \in C^2((0, T) \times \Omega) \cap C^0([0, T] \times \overline{\Omega}) \), prove that
\[
u(t, x) \geq 0, \quad \forall (t, x) \in [0, T] \times \Omega.
\]