Autumn 2019 Math 61CM: Homework
Assignment 10 (Do not turn in!)

1. (a) Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by
\[
f(x, y) = \begin{cases} 1 & \text{if } x = 0, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}
\]
Prove that \( f \) is integrable.

(b) Let \( g : [0, 1] \to \mathbb{R} \) be defined by \( g(y) = f(0, y) \). Prove that \( g \) is not integrable.

2. Let \( R \) be a rectangle in \( \mathbb{R}^n \) and \( A \subseteq R \) be measurable with \( \text{Vol}(A) = 0 \).

(a) Prove that if \( B \subseteq A \), then \( B \) is measurable and that \( \text{Vol}(B) = 0 \).

(b) If \( f : R \to \mathbb{R} \) is an integrable function, prove that \( \int_A f = 0 \).

3. Let \( f, g \) be integrable. Prove the following:

(a) \( \max\{f, g\} \) is integrable.

(b) \( fg \) is integrable.

[Here are some hints. For part 1, first show that for every \( i \),
\[
\sup_{x \in R_i} \max\{f(x), g(x)\} - \inf_{x \in R_i} \max\{f(x), g(x)\} \\
\leq \max\{ \sup_{x \in R_i} f(x) - \inf_{x \in R_i} f(x), \sup_{x \in R_i} g(x) - \inf_{x \in R_i} g(x) \}.
\]
For part 2, note that since \( fg = f_+g_+ + f_-g_- - f_+g_- - f_-g_+ \), by part 1, it suffices to check that if \( f, g \geq 0 \) are integrable, then \( fg \) are integrable.

Now for \( R_k \) a subrectangle of a partition and a bounded function \( h \), denote \( M_i(h) = \sup_{x \in R_i} h(x) \) and \( m_i(h) = \inf_{x \in R_i} h(x) \). Note that if \( f \) and \( g \) are non-negative, then \( M_i(fg) \leq M_i(f)M_i(g) \) and \( m_i(fg) \geq m_i(f)m_i(g) \).]

4. Let \( R \subseteq \mathbb{R}^n \) be a rectangle and \( f, g : R \to \mathbb{R} \) be integrable functions.
Assume that \( f = g \) except on a set of zero volume. Prove that \( \int_R f = \int_R g \).

5. Let \( H \) be the parallelogram in \( \mathbb{R}^2 \) whose vertices are \((1,1), (3,2), (4,5)\) and \((2,4)\).

(a) Find an affine map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) (i.e. \( T(x, y) = z + A \begin{bmatrix} x \\ y \end{bmatrix} \)) for some \( z \in \mathbb{R}^2 \) and some \((n \times n)\) matrix \( A \) which sends \((0,0)\) to \((1,1), (1,0)\) to \((3,2), (0,1)\) to \((2,4)\).

(b) Using the previous part, evaluate
\[
\int_H e^{x+y} \, dx \, dy.
\]