1. Prove for any $x, y \in \mathbb{R}^n$ that $x \cdot y = 0$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{R}$.

2. Let $A = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$. Prove that $A$ preserves the dot product: for any $x, y \in \mathbb{R}^2$, $(Ax) \cdot (Ay) = x \cdot y$.

3. Find an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^2$, but $f$ is not a linear map.

4. Find a $4 \times 4$ matrix over $\mathbb{R}$ such that its null space is the same as its column space.

5. Simon 2.1.2

6. Let $(X, d)$ be a metric space.

   (a) Suppose $U_1, \ldots, U_N$ are open. Show that $\bigcap_{i=1}^N U_i$ is open.

   (b) Suppose $K_1, \ldots, K_N$ are closed. Show using the definition of closedness (i.e. without using the previous part) that $\bigcup_{i=1}^N K_i$ is closed.

7. Simon 2.2.1 [Here, you can assume $m = 1$.]

8. (Bonus problem) Prove or disprove: For any 3 subspaces $U_1, U_2, U_3 \subset \mathbb{R}^n$,
   \[
   \dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3).
   \]