Autumn 2019 Math 61CM: Homework
Assignment 9 (Due 12/3 during lecture)

You are strongly encouraged to work with fellow students on the problems but (1) please write down the names of students whom you discussed with and (2) you must write up your own solutions. Please do not search for solutions on the internet or use online forums.

1. Simon 2.9.1

2. Suppose \( f(x, y, z) = 3xy + z^3 + 3z^2 - 2 \) for \((x, y, z) \in \mathbb{R}^3\).
   
   (a) Prove that \( f|S^2 \) attains its max and min, where \( S^2 \) is the sphere \( \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \).
   
   (b) Find all the (6) critical points of \( f|S^2 \), and hence find the points where \( f|S^2 \) attains its max and min.

3. Simon 2.9.4 [Do this problem directly from the definition of submanifolds, i.e. do not use Theorem 9.13.]

4. Simon A.5.1

5. Simon A.6.1

6. Simon A.6.2

7. Simon A.6.4

8. (Weierstrass M-test) Suppose that \( M_n > 0, \sum_{n=1}^{\infty} M_n \) converges, \( g_n \in C([a, b]) \) and \( |g_n(x)| \leq M_n \) for all \( x \in [a, b] \).
   
   (a) Show that the limit \( \lim_{k \to +\infty} \sum_{n=1}^{k} g_n(x) \) exists for all \( x \in [a, b] \).
   
   Define \( g(x) = \lim_{k \to +\infty} \sum_{n=1}^{k} g_n(x) \).
   
   (b) Show that \( \sum_{n=1}^{\infty} g_n \) converges uniformly to \( g \), i.e.
   
   \[ \lim_{k \to \infty} \sup_{[a, b]} \left| \sum_{n=1}^{k} g_n - g \right| = 0. \]

   (c) Suppose \( f_n \in C([a, b]) \) and there exists a function \( f : [a, b] \to \mathbb{R} \) such that \( f_n \) converges to \( f \) uniformly (i.e. \( \lim_{n \to \infty} \sup_{[a, b]} |f_n - f| = 0 \)).
   
   Prove that \( f \) is continuous. Conclude that in fact for \( g \) as in the previous part, it holds that \( g \in C([a, b]) \).