Unless otherwise indicated, you can use results covered in lecture and homework, provided they are clearly stated.

If necessary, continue solutions on backs of pages.

You have three hours in total.

I understand and accept the provisions of the honor code (Signed) __________________________
1(a) (2 points): Let $Q(\xi) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} \xi_i \xi_j$ be a quadratic form. Prove that there exists $C \in \mathbb{R}$ such that

$$Q(\xi) \leq C \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n$$

and moreover there exists $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ such that $\frac{Q(\xi_0)}{\|\xi_0\|^2} = C$.

1(b) (3 points) Let $A$ be an $(n \times n)$ real symmetric matrix. Prove directly (i.e. without using the spectral theorem) that $A$ has a real eigenvalue. [Hint: use part (a).]
2(a) (3 points): If $V$ is a subspace of $\mathbb{R}^n$ define the orthogonal projection $P$ of $\mathbb{R}^n$ onto $V$, and prove that $\|x - P(x)\| = \min_{y \in V} \|x - y\|$ for each $x \in \mathbb{R}^n$.

2(b) (2 points): Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 2, 0)^T, (1, 0, 0, 3)^T, (0, 2, 0, 1)^T$, and write down an explicit formula (involving numbers and matrix operations only) for the matrix of the orthogonal projection to this subspace (but you do not need to compute it).
3 (a) (3 points): Suppose $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$. (i) Give the proof that $Ax = b$ has at least one solution $x \in \mathbb{R}^n \iff b \in \text{C}(A)$, and (ii) In case $m = n$ and $\text{N}(A) = \{0\}$, prove that $Ax = b$ has a solution for each $b \in \mathbb{R}^n$.

Hint for (ii): Start by applying the rank/nullity theorem.

3 (b) (2 points): If $V$ is a subspace of $\mathbb{R}^n$, give the definition of $V^\perp$. Prove that $V \cap V^\perp = \{0\}$. 