Section problems for Math 61CM, week 5a, 10/21/19

1. Let $V$ be a real vector space (equipped with an inner product) and $W$ be a subspace. Let $P$ denote the orthogonal projection from $V$ onto $W$. Let $x \in V$ and $y \in W$. Show that

$$\frac{d}{dt} \|x - Px + ty\|^2 \bigg|_{t=0} = 0.$$ 

Interpret this in terms of the geometry.

2. Let $(x_n)$ be a sequence in $\mathbb{R}$ so that for every $n \in \mathbb{N}$, we have $|x_n| < \pi/2$. Let $f : (-\pi/2, \pi/2) \to \mathbb{R}$ be a continuous function. Recall that by the Bolzano–Weierstrass theorem, $(x_n)$ has a convergent subsequence. Must $(f(x_n))$ have a convergent subsequence?

3. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of $\mathbb{R}^n$. Let

$$U(A) = \left\{ x \in \mathbb{R}^n \left| \sup_{n \in \mathbb{N}} \|A^n x\| = \infty \right. \right\}.$$ 

Prove that either $U(A) = \emptyset$ or any closed set containing $U(A)$ must be $\mathbb{R}^n$. [Hint: what can you say about $\mathbb{R}^n \setminus U(A)$?]

- If instead we take $A$ to be a nonlinear map, for example $A : \mathbb{C} \to \mathbb{C}$ given by $A(z) = z^2 + c$ for some fixed $c \in \mathbb{C}$, then $U(A)$ can have a much more interesting structure. See Figure 0.1 for example. Do a Google Images search for “Julia sets” if you want to see more; there are many open math problems about the structure of these sets!

4. (Dini’s theorem) Let $(f_n)$ be a sequence of continuous functions $[0, 1] \to \mathbb{R}$. Suppose that, for each $x \in [0, 1]$ and each $n \in \mathbb{N}$, we have $f_n(x) \leq f_{n+1}(x)$. Show that if there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that for each $x \in [0, 1], \lim_{n \to \infty} f_n(x) = f(x)$, then $\lim_{n \to \infty} \sup_{x \in [0,1]} |f(x) - f_n(x)| = 0$. (Note that this works if we replace $[0, 1]$ by any other compact set.)

5. Let $Q$ be equipped with the metric inherited from $\mathbb{R}$. For any $x \neq y \in Q$, find a subset $A$ such that $x \in A$, $y \notin A$, and $A$ is both open and closed in $Q$.

6. Let $X = [0, 1] \cap Q$ be equipped with the metric inherited from $\mathbb{R}$. Find a continuous function $f : X \to \mathbb{R}$ such that there is no continuous function $\tilde{f} : [0, 1] \to \mathbb{R}$ such that $f = \tilde{f}|_X$. 

1
Figure 0.1: $U(A)^c$ for $A(z) = z^2 - 1 + 0.1i$. (Image by Adam Majewski, from commons.wikimedia.org/wiki/File:Filled.jpg.)