Solutions for Math 61CM Practice Midterm 2

1

1(a) Compute

\[
\begin{pmatrix} 1 & 1 & 3 & 0 & -2 \\ -1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 6 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 0 & -2 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 6 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 0 & -2 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 0 & -2 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
\]

1(b) There are two columns in \( \text{rref}(A) \) without pivots, so \( \dim N(A) = 2 \). It is spanned by \((0, -3, 1, 0, 0)\) and \((1, 1, 0, -1, 1)\).

1(c) The columns in the original matrix corresponding to columns in \( \text{rref}(A) \) with pivots form a basis for \( C(A) \), so in this case a basis is given by \{\((1, -1, 0), (1, 0, 2), (0, 1, 1)\)\}. (Note that since \( \dim C(A) = 3 \) by the rank-nullity theorem, any basis of \( \mathbb{R}^5 \) would also work.)

2 See the proof in the book.

3 We use the Gram-Schmidt process. Our first vector is \( \mathbf{w}_1 = (3, 4, 0, 0)/\| (3, 4, 0, 0) \| = (3/5, 4/5, 0, 0) \). Then we subtract the projection onto \( \mathbf{w}_1 \) to get \( \mathbf{w}_2 = (0, 3, 4, 0) - [(0, 3, 4, 0) \cdot (3/5, 4/5, 0, 0)](3/5, 4/5, 0, 0) = (0, 3, 4) - (36/25, 48/25, 0, 0) = (-36/25, 27/25, 4, 0). \) We normalize this vector to get

\[
\mathbf{w}_2 = \frac{(-36/25, 27/25, 4, 0)}{\|(-36/25, 27/25, 4, 0)\|} = \frac{\sqrt{391}}{5/\sqrt{481}}(0, -36/25, 27/25, 4, 0, 0) = \left( \frac{-36}{5 \sqrt{481}}, \frac{27}{5 \sqrt{481}}, \frac{20}{\sqrt{481}} \right).
\]

Then the orthonormal basis is given by \{\( \mathbf{w}_1, \mathbf{w}_2 \)\}.

The matrix is then given by

\[
\begin{pmatrix}
9/25 & 12/25 & 0 & 0 \\
12/25 & 16/25 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1296/12025 & -972/12025 & -144/12025 & 0 \\
-972/12025 & 729/12025 & 481/12025 & 0 \\
-144/12025 & 481/12025 & 481/12025 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
225/481 & 192/481 & -144/481 & 0 \\
192/481 & 337/481 & 481/481 & 0 \\
-144/481 & 481/481 & 481/481 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

4 We compute

\[
\nabla f(x) = \begin{pmatrix} x_1^6 - 1 \\ 6x_1^5 \end{pmatrix}.
\]

This is zero when \( x_1 \in \{\pm 1\} \) and \( x_2 \in \{\pm 2\} \), so the critical points are \((1, 2), (1, -2), (-1, 2), (-1, -2)\).

Then we compute the Hessian matrix

\[
\begin{pmatrix}
6x_1^5 & 0 \\
0 & 6x_2^5
\end{pmatrix}.
\]

A diagonal matrix is positive definite if all of its entries are positive, negative definite if all of its entries are negative, and indefinite if it has both positive and negative entries. So the Hessian matrix is positive definite at \((1, 2)\), negative definite at \((-1, -2)\), and indefinite at \((1, -2)\) and \((-1, 2)\). So by the second derivative test, \((1, 2)\) is a local minimum, \((-1, -2)\) is a local maximum, and \((1, -2)\) and \((-1, 2)\) are neither local maxima nor local minima.
(a) Let \( s_j = \sum_{n=1}^j a_n \). Since \( \sum_{n=1}^{\infty} a_n \) converges, \((s_j)\) converges by definition; call the limit \( s \). Therefore, for every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) so that if \( j \geq N \) then \( |s_j - s| < \varepsilon/2 \). Therefore, if \( j \geq N + 1 \) then \( |a_j| = |s_j - s_{j-1}| \leq |s_j - \ell| + |\ell - s_{j-1}| < \varepsilon \). Thus \((a_j)\) converges to 0.

(b) We note that
\[
\sum_{n=1}^{2N} \frac{1}{n} = 1 + \sum_{j=1}^{N} \left( \sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n} \right) \geq 1 + \sum_{j=1}^{N} \left( \sum_{n=2^{j-1}+1}^{2^j} \frac{1}{2^j} \right) = 1 + \sum_{j=1}^{N} \frac{2^j}{2} = 1 + \frac{N}{2} \to \infty
\]
as \( N \to \infty \), so \( \sum_{n=1}^{2N} \frac{1}{n} \) cannot converge as \( N \to \infty \).

6

(a) I will use boldface to denote vectors, so call the fixed vector \( v \in \mathbb{R}^n \). By the definition of differentiability, we have
\[
\lim_{h \to 0} \frac{f(tv + h) - f(tv) - Df(tv)h}{\|h\|} = 0.
\]
This in particular implies that
\[
\lim_{s \to 0} \frac{f(tv + sv) - f(tv) - Df(tv)sv}{\|sv\|} = 0,
\]
where here the limit is taken for scalars \( s \). This means that
\[
\lim_{s \to 0} \frac{g(t + s) - g(t) - Df(tv)sv}{|s|} = 0,
\]
so also
\[
\lim_{s \to 0} \frac{g(t + s) - g(t) - Df(tv)sv}{s} = 0.
\]
Rearranging, this means that
\[
g'(t) = \lim_{s \to 0} \frac{g(t + s) - g(t)}{s} = Df(tv)v
\]
for all \( t \), and in particular \( g \) is differentiable for all \( t \).

(b) Let \( r(t) = (t, t^2) \). Then \( Dr(t) = (1, 2t) \). Then by the chain rule,
\[
Dq(t) = Dp(r(t)) \cdot Dr(t) = \begin{pmatrix} D_1 p(r(t)) & D_2 p(r(t)) \end{pmatrix} \begin{pmatrix} 1 \\ 2t \end{pmatrix} = D_1 p(r(t)) + 2t D_2 p(r(t)).
\]