Recall first that a series \( \sum_{n=1}^{\infty} a_n \), where \( a_n \in V \), \( V \) a normed vector space, converges if the sequence of partial sums, \( s_k = \sum_{n=1}^{k} a_n \), does, and one writes
\[
\sum_{n=1}^{\infty} a_n = \lim s_k.
\]

Recall also that a series converges absolutely if \( \sum_{n=1}^{\infty} ||a_n|| \) converges; note that this is a real valued series with non-negative terms. If \( a_n \) are real, \( ||a_n|| \) is simply \( |a_n| \), hence the terminology. We then have:

**Theorem 1** If \( V \) is a complete normed vector space, then every absolutely convergent series converges.

**Proof:** Suppose \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent. Since \( V \) is complete, we just need to show that the sequence of partial sums, \( \{s_k\}_{k=1}^{\infty} \), \( s_k = \sum_{n=1}^{k} a_n \), is Cauchy, since by definition of completeness that implies the convergence of \( \{s_k\}_{k=1}^{\infty} \).

But for \( n > m \),
\[
||s_n - s_m|| = \left| \sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j \right| = \sum_{j=m+1}^{n} ||a_j|| \leq \sum_{j=m+1}^{n} ||a_j||.
\]

The right hand side is exactly the difference between the corresponding partial sums of \( \sum_{n=1}^{\infty} ||a_n|| \).

Namely, with \( \sigma_n = \sum_{j=1}^{n} ||a_j|| \), and for \( n > m \), we have
\[
||s_n - s_m|| = ||\sigma_n - \sigma_m|| = \sum_{j=m+1}^{n} ||a_j||,
\]

where we used that \( ||a_j|| \geq 0 \), so the sequence of partial sums is increasing, in order to drop the absolute value. In combination,
\[
||s_n - s_m|| \leq ||\sigma_n - \sigma_m||,
\]
at first when \( n > m \), but the same argument works if \( n < m \) with \( n, m \) interchanged, and if \( n = m \), both sides vanish.

So now to prove that \( \{s_k\}_{k=1}^{\infty} \) is Cauchy, let \( \varepsilon > 0 \). Since \( \{\sigma_k\}_{k=1}^{\infty} \) converges, it is Cauchy, so there exists \( N \in \mathbb{N}^+ \) such that for \( n, m \geq N \), \( ||\sigma_n - \sigma_m|| < \varepsilon \). Then for \( n, m \geq N \), \( ||s_n - s_m|| \leq ||\sigma_n - \sigma_m|| < \varepsilon \), completing the proof. \( \square \)

While the problem set shows that the rearrangement of series that do not converge absolutely leads to many potential consequences (divergence, convergence to a different limit), absolutely convergent series are well-behaved. First:

**Definition 1** A rearrangement of \( \sum_{n=1}^{\infty} a_n \) is a series \( \sum_{n=1}^{\infty} a_{j\left(n\right)} \), where \( j : \mathbb{N}^+ \rightarrow \mathbb{N}^+ \) is a bijection.

Let us consider non-negative series first (such as the norms of the terms of an arbitrary series).

**Theorem 2** Suppose \( a_n \geq 0 \) for all \( n \in \mathbb{N}^+ \), \( a_n \) real. Let \( S \) be the set of all finite sums of the \( a_n \), i.e. the set of all sums \( \sum_{n \in B} a_n \) where \( B \subset \mathbb{N}^+ \) is finite. Then \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( S \) is bounded, and in that case \( \sum_{n=1}^{\infty} a_n = \sup S \).

**Proof:** Let \( s_k = \sum_{n=1}^{k} a_n \) be the kth partial sum, and \( R \) be the set of partial sums \( \{s_k : k \in \mathbb{N}^+ \} \). We already know that the increasing sequence \( \{s_k\}_{k=1}^{\infty} \) converges if and only it is bounded above, i.e. iff \( R \) is bounded above, and in that case \( \lim s_k = \sup R \).
Now $R \subset S$, so if $S$ is bounded above so is $R$, and $\sup R \leq \sup S$ since $\sup S$ is an upper bound for $S$, thus for $R$, and $\sup R$ is the least upper bound.

On the other hand, let $B \subset \mathbb{N}^+$ finite, and let $K = \max B$ (exists because $B$ is finite). Then $s_K = \sum_{n=1}^{K} a_n \geq \sum_{n \in B} a_n$ since $B \subset \{1, 2, \ldots, K\}$ and since $a_n \geq 0$. Thus for all elements $s = \sum_{n \in B} a_n$, $B$ finite, of $S$, there exists $r = r_K \in R$ such that $r \geq s$. Correspondingly, if $R$ is bounded above, then so is $S$, with $\sup R \geq r \geq s$ for all $s \in S$, i.e. $\sup R$ is an upper bound for $S$, so $\sup R \geq \sup S$.

Thus, if either one of $S, R$ is bounded above, so is the other, i.e. both are bounded above, and one has $\sup R \leq \sup S$ as well as $\sup R \geq \sup S$, so the two are equal: $\sup S = \sup R = \sum_{n=1}^{\infty} a_n$. □

As an immediate consequence we have

**Theorem 3** Suppose $a_n \geq 0$ for all $n$, $a_n$ real, and $\sum_{n=1}^{\infty} a_n$ converges. Then any rearrangement $\sum_{n=1}^{\infty} a_{j(n)}$ converges and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.

**Proof:** This is very easy now: let $S$ be the set of all finite sums of terms in the series as above. By the previous theorem, $\sum_{n=1}^{\infty} a_n$ converges implies that $S$ is bounded above and $\sum_{n=1}^{\infty} a_n = \sup S$. But the set of finite sums of terms of the rearranged series is also $S$! Thus, again by the previous theorem, the rearranged series also converges, with $\sum_{n=1}^{\infty} a_{j(n)} = \sup S$. Combining these two proves the theorem. □

This can be used to show that real valued absolutely convergent series can be rearranged: write $a_n = p_n - q_n$ with $p_n, q_n \geq 0$ being the ‘positive part’ and ‘negative part’ as in the text; if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge since $p_n, q_n \leq |a_n|$, but these can be rearranged by the previous theorem, to converge to the same limit, and then $\sum_{n=1}^{\infty} a_{j(n)}$ also converges as $a_{j(n)} = p_{j(n)} - q_{j(n)}$, with

$$
\sum_{n=1}^{\infty} a_{j(n)} = \sum_{n=1}^{\infty} p_{j(n)} - \sum_{n=1}^{\infty} q_{j(n)} = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} a_n.
$$

The general theorem is

**Theorem 4** If $V$ a complete normed vector space, $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} a_{j(n)}$ converges absolutely and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.

**Proof:** We already know that $\sum_{n=1}^{\infty} a_n$ converging absolutely, i.e. $\sum_{n=1}^{\infty} \|a_n\|$ converging, implies $\sum_{n=1}^{\infty} \|a_{j(n)}\|$ converging, i.e. $\sum_{n=1}^{\infty} a_{j(n)}$ converging absolutely (and in particular converging). Thus, the only remaining statement is to show the equality of the sums: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.

The key idea of the proof is that absolute convergence means given any $\varepsilon > 0$ that there are finitely many terms in the series such that if one takes any other finitely many terms, the sum of their norms is $< \varepsilon$.

So let $\varepsilon > 0$. First, since $\sum_{k=1}^{\infty} \|a_k\|$ converges, thus is Cauchy, means that there exists $N_1$ such that $n, m \geq N_1$ implies $|\sigma_n - \sigma_m| < \varepsilon$, where $\sigma_n = \sum_{i=1}^{n} \|a_i\|$. Thus, for $n > m = N_1$,

$$
\sum_{i=N_1+1}^{n} \|a_i\| = |\sigma_n - \sigma_{N_1}| < \varepsilon.
$$

This is exactly the statement that any finitely many of the $a_i$ which do not include $a_1, \ldots, a_{N_1}$ have the sum of their norms $< \varepsilon$. Indeed, suppose $B \subset \mathbb{N}^+$ is finite with all elements $\geq N_1 + 1$. Let $K = \max B$
(finite set, so maximum exists), and observe that for each \( i \in B \), \( i \in \{N_1 + 1, N_1 + 2, \ldots, K\} \). Thus  
\[
\sum_{i \in B} \|a_i\| \leq \sum_{i=N_1+1}^{K} \|a_i\| < \varepsilon.
\]
Hence, by the triangle inequality one also has  
\[
\sum_{i \in B} \|a_i\| \leq \sum_{i \in B} \|a_i\| < \varepsilon.
\]

Now, let \( s = \sum_{n=1}^{\infty} a_n \), resp. \( r = \sum_{n=1}^{\infty} a_{j(n)} \), and let \( \{s_k\}_{k=1}^{\infty} \), resp. \( \{r_k\}_{k=1}^{\infty} \) be sequence of partial sums of the two series. Let \( N_2 = \max A \), \( A = \{j^{-1}(1), \ldots, j^{-1}(N_1)\} \), so for \( n \geq N_2 + 1 \), \( j(n) \notin \{1, \ldots, N_1\} \). Thus, for \( n \geq N = \max\{N_1, N_2\} \), the terms of both \( s_n \) and \( r_n \) include \( a_i \) for all \( i \leq N_1 \). Thus,  
\[
s_n - r_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{j(i)} = \sum_{i=N_1+1}^{n} a_i - \sum_{i \in \{1,\ldots,n\}\setminus A} a_i,
\]
where on the right hand side we dropped \( \sum_{i=1}^{N_1} a_i = \sum_{i \in A} a_{j(i)} \) from both sums whose difference we are taking. But \( \{N_1 + 1, \ldots, n\} \) and \( \{1, \ldots, n\}\setminus A \) are finite sets disjoint from \( \{1, \ldots, N_1\} \). Thus, by the above observation, applied with \( B = \{N_1 + 1, \ldots, n\} \), resp. \( B = \{1, \ldots, n\}\setminus A \)
\[
\left\| \sum_{i=N_1+1}^{n} a_i \right\| < \varepsilon, \quad \left\| \sum_{i \in \{1,\ldots,n\}\setminus A} a_i \right\| < \varepsilon.
\]

We thus conclude that  
\[
\|s_n - r_n\| \leq \left\| \sum_{i=N_1+1}^{n} a_i \right\| + \left\| \sum_{i \in \{1,\ldots,n\}\setminus A} a_i \right\| < 2\varepsilon.
\]

In summary, we have shown that for all \( \varepsilon > 0 \) there exists \( N \) such that for \( n \geq N \), \( |s_n - r_n| < 2\varepsilon \). This shows that \( \lim(s_n - r_n) = 0 \), and thus \( \lim s_n = \lim r_n \), since both sequences of partial sums converge. \( \square \)