NOTES ON (IN)STABILITY OF HYPERBOLIC EQUILIBRIUM POINTS

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In these notes, we give a discussion of the stable manifold theorem. This theorem gives a complete picture of what happens near a hyperbolic equilibrium point. Time will not permit us to give a complete proof of the theorem, and we will just be able to give a somewhat restricted version (cf. Theorem 3.6). This will nonetheless be sufficient to understand the stability and instability of all hyperbolic equilibrium points. In particular, it also implies the “theorem on stability in first approximation” in §23.3 of the textbook. When preparing for these lectures, I have greatly been benefited from previous lecture notes of Brendle, which I follow to a large extent. As always, this is a preliminary version: if you have any comments or corrections, even very minor ones, please send them to me.

1. Example

Our goal in these notes is to try to understand how by studying the linearized ODE at an equilibrium point, one can essentially determine the behavior of the solutions with initial data close to this equilibrium point. In particular, one can understand the stability (and asymptotic stability) of the equilibrium point.

To get an idea of what is to come, it is useful to have the following example in mind. Consider the ODE

\[ x'(t) = x(t) + y^2(t), \quad y'(t) = -y(t), \]

with initial data

\[ x(0) = x_0, \quad y(0) = y_0. \]

Clearly, \((0, 0)\) is an equilibrium solution. Let us notice that the “linear part” of the ODE (we will make this precise soon) is

\[ x'(t) = x(t), \quad y'(t) = -y(t). \]

From what we discussed earlier in the course, the dynamics for this linear system can be completely understood. Namely, since there is exactly one positive eigenvalue and exactly one negative eigenvalue, we know that \(\mathbb{R}^n\) can be decomposed into the stable and unstable subspaces. For any initial data on the stable subspace, the solution converges to 0; while for any initial data not on the stable subspace, the solution grows without bound.

Now, let’s proceed to solve the nonlinear ODE above. First,

\[ y(t) = y_0 e^{-t}. \]

Plug this into the other equation, we get \(x'(t) = x(t) + y_0^2 e^{-2t}\), which is an inhomogeneous linear equation. The solution is given

\[ x(t) = x_0 e^t + \frac{y_0^2}{3} e^{t - 3t} \left(\frac{1}{3} e^{-3t} - \frac{1}{3}\right) = (x_0 - \frac{1}{3} y_0^2) e^t + \frac{1}{3} y_0^2 e^{-2t}. \]

Observe that the solution decays as \(t \to \infty\) if and only if \(x_0 - \frac{1}{3} y_0^2 = 0\). In a sense the picture is “quite similar” to the picture in the linear case. There is a subset \(\{(x_0, y_0) : x_0 - \frac{1}{3} y_0^2 = 0\}\) – which is no longer a subspace of \(\mathbb{R}^2\) – such that any initial data on this set give rise to solutions which converge to 0 and any initial data not on this set give rise to solutions which “move away from 0”. While this subset is not a subspace, it is nonetheless a manifold. We will call this set the stable manifold.

We will show that in fact given any nonlinear ODE, if the linearized ODE at a equilibrium point is hyperbolic (a notion that we will define below), then this is always the picture locally near the equilibrium. We will make this precise in the upcoming sections.

\(^1\)Of course, all mistakes, misprints, etc, are my own.
2. Setup and definition of hyperbolic equilibrium

We will consider stability of equilibrium solutions to nonlinear autonomous ODEs

\[ u'(t) = F(u(t)), \]  

where \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is \( C^1 \). In particular, by Picard–Lindelöf theorem, we know that given any initial data, there always exists a local-in-time solution.

Suppose we have an equilibrium solution to (2.1). Let us assume without loss of generality (translate \( u \mapsto u - c \) otherwise) that the equilibrium in question is \( u(t) = 0 \). Since \( F \) is \( C^1 \), we can write

\[ u'(t) = Au(t) + G(u(t)), \]

where \( A \) is an \((n \times n)\) real matrix and \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a \( C^1 \) function satisfying \( \lim_{\|x\| \rightarrow 0} \|G(x)\| = 0 \).

**Definition 2.1.** Given the setting as above, we say that 0 is a hyperbolic equilibrium point if \( A \) has no eigenvalues on the imaginary axis.

From now on, we will assume that we have a hyperbolic equilibrium point, and use notations as above, even if we do not state this explicitly in the statement of the results. The main result we will prove is that locally near a hyperbolic equilibrium point, the picture “looks like” the corresponding picture in the linear case.

3. The stable manifold theorem

Before we go further, let us set up some notations. Recall that we can decompose \( \mathbb{R}^n \) according to the eigenspaces of \( A \) as follows:

\[ \mathbb{R}^n = \mathbb{R} P_+ \oplus \mathbb{R} P_-, \]

where \( \mathbb{R} P_+ \) (resp. \( \mathbb{R} P_- \)) is the direct sum of all the generalized eigenspaces of \( A \) corresponding to eigenvalues with positive (resp. negative) real parts. Define \( \pi_{\pm} \) to be the corresponding the projection maps

\[ \pi_{\pm} : \mathbb{R}^n \rightarrow \mathbb{R} P_{\pm}. \]

Let us note that \( \pi_{\pm} \) commutes with \( A, e^{tA}, \) etc. Since \( A \) has no eigenvalues on the imaginary axis, there exist \( \Lambda > 0 \) and \( \alpha > 0 \) such that

\[ \|e^{t\pi_-}\|_{op} \leq \Lambda e^{-\alpha t}, \quad t \geq 0, \]

\[ \|e^{t\pi_+}\|_{op} \leq \Lambda e^{\alpha t}, \quad t \leq 0. \]

Our main lemma is the following:

**Lemma 3.1.** Given setup as above, there exists \( r > 0 \) sufficiently small such that the following holds: Given \( u_- \in R P_- \cap B(0, r) \), there exists a unique function \( u(t) \) such that

- \( u(t) \in B(0, 2\Lambda r) \) for all \( t \geq 0 \), and
- \[ u(t) = e^{At}u_- + \int_0^t e^{(t-s)A} \pi_- G(u(s)) \, ds - \int_t^\infty e^{(t-s)A} \pi_+ G(u(s)) \, ds. \]

Moreover,

- \( u(t) \) satisfies \( u'(t) = Au(t) + G(u(t)) \) with initial data \( u(0) = u_- - \int_0^\infty e^{(t-s)A} \pi_+ G(u(s)) \, ds \), and
- \( \|u(t)\| \leq 2\Lambda \|u_-\|e^{-\frac{t}{r}}. \)

Let us postpone the proof of the lemma and look at the consequences. Let us fix \( r > 0 \) so that the lemma holds. The above lemma allows us to define a map \( \psi : \mathbb{R} P_- \cap B(0, r) \rightarrow \mathbb{R} P_+ \) by taking

\[ \psi(u_-) = -\int_0^\infty e^{(t-s)A} \pi_+ G(u(s)) \, ds, \]

where \( u(t) \) is the function as given in Lemma 3.1.

We now define the stable manifold:

\(^2\)One can also restrict the domain of \( F \) to an open set \( U \subset \mathbb{R}^n \).
Definition 3.2. We define
\[ W^s := \{ u_- + \psi(u_-) : u_- \in B(0, r) \} \]
to be the stable manifold.

We can now establish that picture as we mentioned in Section 1. First we show that the stable manifold is “stable” in the following sense:

Corollary 3.3. Consider initial data \( u_0 \in W^s \). Then, the unique solution to the ODE (2.1) with initial data \( u(0) = u_0 \) converges to 0. Moreover, it holds that
\[ \| u(t) \| \leq 2\Lambda \| u_- \| e^{-\frac{\alpha t}{2}}. \]

Proof. This is a direct consequence of Lemma 3.1. \( \square \)

This implies the following stability theorem:

Corollary 3.4 (Theorem on stability in first approximation). If \( R^+ P_+ = \{0\} \), then 0 is an asymptotically stable equilibrium to (2.1).

Proof. This is immediate from Corollary 3.3 and the fact that in this case \( W^s = B(0, r) \). \( \square \)

Conversely, we show that

Corollary 3.5. Let \( u_0 = u_- + u_+ \), where \( u_- \in \mathbb{R}^+ P_- \cap B(0, r) \) and \( u_0 \in B(0, 2\Lambda r) \). If \( u_0 \notin W^s \), then the unique solution to the ODE (2.1) leaves the ball \( B(0, 2\Lambda r) \) in finite positive time.

Proof. Suppose not, i.e., assume that the solution remains in \( B(0, 2\Lambda r) \) for all \( t \geq 0 \). Writing the solution as
\[ u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}G(u(s)) \, ds, \tag{3.1} \]
applying \( e^{-At}\pi_+ \) we have
\[ e^{-At}\pi_+u(t) = \pi_+u_0 + \int_0^t e^{-As}\pi_+G(u(s)) \, ds \]
Using that \( u(t) \) is bounded and taking \( t \to \infty \), we have
\[ 0 = u_+ + \int_0^t e^{-As}\pi_+G(u(s)) \, ds. \tag{3.2} \]
Subtracting \( (e^{At} \times (3.2)) \) from (3.1), we have
\[ u(t) = e^{At}u_- + \int_0^t e^{A(t-s)}\pi_-G(u(s)) \, ds - \int_t^\infty e^{A(t-s)}\pi_+G(u(s)) \, ds. \]
But then by the uniqueness statement in Lemma 3.1 and the definition of \( \psi \), we have \( u_+ = \psi(u_-) \). This implies \( u_0 \in W^s \) and contradicts our assumption. \( \square \)

Let us summarize again the results above in the following theorem:

Theorem 3.6 (Stable manifold theorem). Given the setting as above, \( W^s \) has the following properties

- If the initial data \( u_0 \in W^s \), then the unique solution to the ODE (2.1) converges to 0. Moreover, it holds that
\[ \| u(t) \| \leq 2\Lambda \| u_- \| e^{-\frac{\alpha t}{2}}. \]
- If the initial data \( u_0 \in B(0, 2\Lambda r) \setminus W^s \) with \( \pi_-u_0 \in B(0, r) \), then the unique solution to the ODE (2.1) leaves the ball \( B(0, 2\Lambda r) \) in finite positive time.
4. Proof of the main lemma

We now prove Lemma 3.1. Let’s have a preliminary observation. For every \( \epsilon > 0 \), there exists \( r > 0 \) such that if \( y, z \in B(0, 2r) \), we have

\[
\|G(y) - G(z)\| \leq \epsilon \|y - z\|. \tag{4.1}
\]

This is a consequence of the fact that the derivative of \( G \) at 0 is 0 and that \( G \) is \( C^1 \). We will fix \( \epsilon > 0 \) at the end. The \( r \) (as in the statement of Lemma 3.1) will be chosen so that (4.1) holds.

We first prove the existence part of Lemma 3.1:

**Proof of Lemma 3.1 (Existence part).** The lemma will be proven by an iteration. Let \( \epsilon > 0 \) be a constant to be chosen and take \( r > 0 \) such that (4.1) holds. Let \( u_\infty \in \mathbb{R} P_- \cap B(0, r) \). Define \( u_0 = 0 \). Given \( u_{k-1} : \mathbb{R} \to B(0, 2r) \) (for some \( k \geq 1 \)), define \( u_k : \mathbb{R} \to B(0, 2r) \) by

\[
u_k(t) = e^{At}u_\infty + \int_0^t e^{(t-s)A}\pi_- G(u_{k-1}(s)) \, ds - \int_t^\infty e^{(t-s)A}\pi_+ G(u_{k-1}(s)) \, ds.
\]

We will show that \( u_k(t) \) has a uniform limit \( u(t) \), which satisfies the desired properties as stated in the lemma.

**\( u_k(t) \) is Cauchy.** We will inductively prove that

- \( u_k \) is well-defined, i.e., \( \|u_k\| < 2\Lambda r \), and
- the following quantitative estimate

\[
\|u_k(t) - u_{k-1}(t)\| \leq 2^{-k+1}\Lambda\|u_\infty\|e^{-\alpha t/2}.
\]

holds for all \( t \geq 0 \) and for all \( k \geq 1 \).

We first check the base case. Since \( G(0) = 0 \), \( u_1(t) = e^{At}u_\infty \) satisfies the estimate

\[
\|u_1(t) - u_0(t)\| = \|u_1(t)\| = \|e^{At}u_\infty\| \leq \Lambda e^{-\alpha t}\|u_\infty\| \leq \Lambda\|u_\infty\|e^{-\alpha t/2}.
\]

In particular, \( \|u_1(t)\| < 2\Lambda r \), and \( u_1(t) \) is well-defined.

We now carry out the inductive step. Assume that for some \( k \geq 1 \) that

\[
\|u_j(t) - u_{j-1}(t)\| \leq 2^{-j+1}\Lambda\|u_\infty\|e^{-\alpha t/2}
\]

holds for all \( t \geq 0 \) and for all \( j \leq k \). Now, we estimate

\[
\|u_{k+1}(t) - u_k(t)\|
\leq \|\int_0^t e^{(t-s)A}\pi_- [G(u_k(s)) - G(u_{k-1}(s))] \, ds\| + \|\int_t^\infty e^{(t-s)A}\pi_+ [G(u_k(s)) - G(u_{k-1}(s))] \, ds\|
\leq \int_0^t \|e^{(t-s)A}\pi_-\| \|G(u_k(s)) - G(u_{k-1}(s))\| \, ds + \int_t^\infty \|e^{(t-s)A}\pi_+\| \|G(u_k(s)) - G(u_{k-1}(s))\| \, ds
\leq \epsilon \Lambda \int_0^t e^{-\alpha(t-s)}\|u_k(s) - u_{k-1}(s)\| \, ds + \epsilon \Lambda \int_t^\infty e^{\alpha(t-s)}\|u_k(s) - u_{k-1}(s)\| \, ds
\leq \epsilon \Lambda^2 2^{-k+1}\|u_\infty\| \left( \int_0^t e^{-\alpha(t-s)}e^{-\alpha s/2} \, ds + \int_t^\infty e^{\alpha(t-s)}e^{-\alpha s/2} \, ds \right)
\leq \epsilon \Lambda^2 2^{-k+1}\|u_\infty\|(2e^{-\alpha t/2}).
\]

Choosing \( \epsilon < \frac{\alpha}{4\Lambda} \), the RHS is \( \leq \Lambda 2^{-(k+1)+1}\|u_\infty\|e^{-\alpha t/2} \), as we wanted. Using this, we now compute that

\[
\|u_{k+1}(t)\| \leq \sum_{j=1}^{k+1} \|u_j(t) - u_{j-1}(t)\| \leq \Lambda \|u_\infty\| 2^{-j+1} e^{-\alpha t/2} < 2\Lambda r.
\]

Hence, \( u_{k+1}(t) \) is well-defined.

**Convergence of \( u_k(t) \).** The estimates we showed above implies that \( u_k(t) \) is a Cauchy sequence for each \( t \geq 0 \). Hence, we can define

\[
u(t) = \lim_{k \to \infty} u_k(t).
\]

It is easy to check that \( u(t) \) satisfies all the desired properties in Lemma 3.1. \( \square \)
Next, we prove uniqueness:

**Proof of Lemma 3.1 (Uniqueness part).** Let $\epsilon > 0$ be a constant to be chosen and take $r > 0$ such that (4.1) holds. Suppose $u(t)$ and $v(t)$ are two functions satisfying the conclusion of the lemma. We bounded the difference $u(t) - v(t)$ as follows:

$$
\|u(t) - v(t)\| \\
\leq \|\int_0^t e^{A(t-s)}\pi_-(G(u(s)) - G(v(s))) \, ds\| + \|\int_t^\infty e^{A(t-s)}\pi_+(G(u(s)) - G(v(s))) \, ds\| \\
\leq \int_0^t \|e^{A(t-s)}\pi_\|_{op}\|G(u(s)) - G(v(s))\| \, ds + \int_t^\infty \|e^{A(t-s)}\pi_\|_{op}\|G(u(s)) - G(v(s))\| \, ds \\
\leq \epsilon \Lambda \int_0^t e^{-\alpha(t-s)}\|u(s) - v(s)\| \, ds + \epsilon \Lambda \int_t^\infty e^{\alpha(t-s)}\|u(s) - v(s)\| \, ds \\
\leq \epsilon \Lambda \frac{2}{\alpha} \sup_{s \in [0, \infty)} \|u(s) - v(s)\|.
$$

Since this holds for all $t \in [0, \infty)$, by choosing $\epsilon < \frac{\alpha}{4\Lambda}$, we have

$$
\sup_{t \in [0, \infty)} \|u(t) - v(t)\| \leq \frac{1}{2} \sup_{t \in [0, \infty)} \|u(t) - v(t)\|,
$$

which obviously implies that $u(t) = v(t)$ for all $t \in [0, \infty)$.

At this point, we can fix $\epsilon$, and therefore also $r > 0$. This completes the proof of Lemma 3.1. \qed

5. Final remarks

You may be curious as to why $W^s$ is called a stable manifold, in particular because we have not proven that it is a manifold! In fact, $W^s$ is indeed a $C^1$ manifold. Alternatively, this means that $\psi$ is a $C^1$ map. Moreover, the manifold passes through the origin, and the tangent space at the origin is $\mathbb{R}P^-$. Although these statements are not too much harder than those we have proven, we will not be able to cover them in this course due to time constraints.