1. (5 points) Find the solution to the differential equation \( u'(t) = -\frac{2t}{1+t^2}u(t) + 1 \) with initial condition \( u(0) = 1 \).

2. (5 points) Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a smooth function. Suppose \( u : (-1, 1) \rightarrow \mathbb{R}^n \) is a continuous function such that for every \( t \in (-1, 1) \), the following holds:
\[
    u(t) = u_0 + \int_0^t F(u(\tau)) \, d\tau.
\]
Show that \( u \) is differentiable and satisfies:
\[
    u'(t) = F(u(t)), \quad u(0) = u_0
\]
for \( t \in (-1, 1) \).

3. (10 points) Given an ODE
\[
u'(t) = F(u(t)), \quad u(0) = u_0,\]
define
\[
T_- := \inf\{T : \text{a solution } u(t) \text{ exists for } t \in (T, 0]\}
\]
and
\[
T_+ := \sup\{T : \text{a solution } u(t) \text{ exists for } t \in [0, T])\}.
\]
(a) Consider the ODE for \( u : I \subset \mathbb{R} \rightarrow \mathbb{R} \).
\[
    u'(t) = (u(t))^2, \quad u(0) = u_0.
\]
For each \( u_0 \), find \( T_- \) and \( T_+ \). Conclude that the ODE has a global solution (i.e., a solution \( u : \mathbb{R} \rightarrow \mathbb{R} \)) if and only if \( u_0 = 0 \).
(b) Find a smooth function \( F : \mathbb{R} \rightarrow \mathbb{R} \) such that for some \( u_0 \in \mathbb{R} \), the solution to
\[
    u'(t) = F(u(t)), \quad u(0) = u_0
\]
has \( T_- \), \( T_+ \) both finite.

4. (10 points, cf. Problem 1 on p. 54) Find a periodic solution to \( u'(t) = -u(t) + \sin t \). Is the solution you found stable? Justify your answer.

5. (10 points) Consider linear equations of the form
\[
u'(t) = a(t)u(t) + f(t)
\]
with \( a : \mathbb{R} \rightarrow \mathbb{R}, f : \mathbb{R} \rightarrow \mathbb{R} \) smooth.
(a) Give an example of \( a \) and \( f \) which are both periodic of period \( T \) such that there are no periodic solutions of period \( T \). Justify your answer.
(b) Give an example of \( a \) and \( f \) which are both periodic of period \( T \) such that there are infinitely many distinct periodic solutions of period \( T \). Justify your answer.

6. (10 points) Let \( I \subset \mathbb{R} \) be an open interval. Suppose \( F : I \to \mathbb{R} \) is a bounded function and suppose there exists \( C > 0 \) such that for every \( x, y \in I \),
\[
|F(x) - F(y)| \leq C|x - y|\log|x - y|.
\]
Consider now the initial value problem
\[
\begin{align*}
u'(t) &= F(u(t)), & u(0) &= u_0,
\end{align*}
\]
where \( F(u_0) = 0 \) for some \( u_0 \in I \). Show that \( u(t) = u_0 \) is the unique solution to the initial value problem above.

7. (10 points) Consider the initial value problem
\[
\begin{align*}
u'(t) &= v(t), & v'(t) &= -4u(t), & u(0) &= 0, & v(0) &= 2.
\end{align*}
\]
(a) Show, by explicit computation, that \( u(t) = \sin 2t, v(t) = 2\cos 2t \) is a solution.
(b) Consider the Picard’s iteration (i.e., \( u_0(t) = 0, v_0(t) = 2 \) and \( u_k(t) = \int_0^t v_{k-1}(s) \, ds, v_k(t) = 2 - 4\int_0^t u_{k-1}(s) \, ds \) for \( k \geq 1 \)). Show explicitly that there exists \( \epsilon > 0 \) such that for \( |t| < \epsilon \), \( u_k(t) \to \sin 2t \), \( v_k(t) \to 2\cos 2t \) as \( k \to \infty \). [Hint: Compare \( u_k(t) \) with the Taylor’s series of \( \sin 2t \) around \( t = 0 \). You may use any results from 61CM provided they are clearly stated.]

8. (Bonus problem, 10 points) Show that
\[
u'(t) = \sqrt{u(t)}, \quad u(0) = 0
\]
has infinitely many distinct solutions \( u : [0, 1] \to \mathbb{R} \).