Spring 2017 Math 63CM: Homework
Assignment 2 (Due 4/24 during lecture)

1. (a) (5 points) Construct a sequence of functions \( \{ f_n \}_{n=1}^{\infty} \) with \( f_n : [0, 1] \to \mathbb{R} \) being continuous functions for all \( n \) such that \( \{ f_n \}_{n=1}^{\infty} \) is uniformly bounded but there is no subsequence of \( f_n \) which converges uniformly to a continuous limit.

(b) (5 points) Show explicitly that the sequence \( \{ f_n \}_{n=1}^{\infty} \) you constructed is not equicontinuous.

2. (a) (10 points) Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be smooth. Suppose there is a \( C > 0 \) such that for every \( x \in \mathbb{R}^n \),

\[
\| F(x) \| \leq C \| x \|. \tag{1}
\]

Show that any solution to \( u'(t) = F(u(t)) \), \( u(0) = u_0 \) can be extended for all time, i.e., it is defined on all of \( \mathbb{R} \).

(b) (5 points) Show by an explicit example that the conclusion in part (a) does not hold if (1) is replaced by

\[
\| F(x) \| \leq C \| x \|^2.
\]

3. (5 points) Consider the Newton’s equation describing the motion of one particle

\[
y''(t) = - (\nabla V)(y(t))
\]

for some smooth \( V : \mathbb{R}^3 \to \mathbb{R} \). Denoting \( V = V(x_1, x_2, x_3) \), suppose

\[
x_1 \frac{\partial V}{\partial x_2} - x_2 \frac{\partial V}{\partial x_1} = 0.
\]

Show that for any solution \( y(t) = (y_1(t), y_2(t), y_3(t)) \), the function \( y_1(t)y_2'(t) - y_1'(t)y_2(t) \) is independent of \( t \).

4. (10 points) Consider the Newton’s equation \( y''(t) = - (\nabla V)(y(t)) \) for some smooth \( V : \mathbb{R}^n \to \mathbb{R} \). Suppose \( y(t) \) is a solution whose maximal interval of existence is \( (-\infty, T) \), where \( T < \infty \). Show that there exists a sequence \( t_n \to T \) such that \( V(y(t_n)) \to -\infty \).

5. (10 points) Consider the equation \( u'(t) = - (\nabla V)(u(t)) \) for some smooth \( V : \mathbb{R}^n \to \mathbb{R} \). Suppose \( \bar{u} \in \mathbb{R}^n \) is an equilibrium point such that \( V \) attains a strict local minimum at \( \bar{u} \). Prove that \( \bar{u} \) is stable.