1. We assume without loss of generality that $a_n \neq 0$. Let us define $n$ variables by $u_1 = y, u_2 = y', ..., u_n = y^{(n-1)}$. It is easy to check that these satisfy the first-order linear system $u'(t) = Au(t)$ where

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \ldots & -\frac{a_{n-1}}{a_n} \end{pmatrix}$$

We see then that

$$\chi_A(\lambda) = \det(\lambda I - A) = \frac{1}{a_n} (a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0)$$

The Jordan normal form of $A$ will consist of Jordan blocks with eigenvalues the distinct roots $\lambda_1, ..., \lambda_m$ of the polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$. Therefore the first coordinate of $u(t)$, which is $y(t)$, will generally be expressible in the form

$$y(t) = \sum_{j=1}^{m} p_j(t)e^{\lambda_j t}$$

where $p_j(t)$ is a polynomial of degree less than the multiplicity $\nu_j$ of the root $\lambda_j$ in $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$.

2. We give pictures of the phase curves and the vector fields in each case. For parts (b) and (c), the phase curves lie in two-dimensional planes, so we give pictures of the phase curves in these planes instead.

(a)
3. We know that $A, B$ are similar to the rotation matrices

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad R_\beta = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

respectively. Then the two systems are topologically equivalent if and only if $u'(t) = R_\alpha u(t)$ and $u'(t) = R_\beta u(t)$ are.

The flow of the former is $\phi_t(u) = e^{R_\alpha t}u = R_\alpha t u$ and of the latter $\psi_t(u) = R_\beta t u$. Suppose that the two systems are topologically equivalent via the homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then we have

$$f(\phi_t(u)) = \psi_t(f(u)) \iff f(R_\alpha t u) = R_\beta t f(u)$$

Then for $t = \frac{2\pi}{\alpha}$, we obtain $f(u) = R_{2\pi\beta/\alpha} f(u)$ for all $u$ and thus $R_{2\pi\beta/\alpha} = I \iff \frac{\beta}{\alpha} \in \mathbb{Z}$. By the same argument, setting $t = \frac{2\pi}{\beta}$, we get $\frac{\alpha}{\beta} \in \mathbb{Z}$. We conclude that we must have $\alpha = \pm \beta$. Both are possible, since we can easily
check that taking \( f = Id \) when \( \alpha = \beta \) or \( f \) to be the reflection along the \( x \)-axis when \( \alpha = -\beta \) we indeed get topological equivalences.

4. Suppose that \( f \) gives a topological equivalence between the two systems. Let \( \phi_t \) be the flow of \( u'(t) = Au(t) \) and \( \psi_t \) the flow of \( u'(t) = Bu(t) \). Then \( f(\phi_t(u)) = \psi_t(f(u)) \iff f(e^{At}u) = e^{Bt}f(u) \). In particular, for \( u = 0 \) we get \( f(0) = e^{Bt}f(0) \Rightarrow f(0) = 0 \) (the eigenvalues of \( e^{Bt} \) are \( e^\lambda \), where \( \lambda \) is an eigenvalue of \( B \), and therefore cannot be equal to 1). Then if \( u \) is in the stable subspace \( V_- \cong \mathbb{R} \) of the first system we have \( e^{At}u \to 0 \) as \( t \to +\infty \) and thus \( e^{Bt}f(u) \to f(0) = 0 \) as \( t \to +\infty \), i.e. \( f(u) \) lies in the stable subspace \( W_- \cong \mathbb{R}^2 \) of the second system. The same argument works for the inverse of \( f \), which in particular yields a continuous, injective map \( W_- \cong \mathbb{R}^2 \not\to \mathbb{R} \cong V_- \).

We show that such a map cannot exist. Consider the \( x \)-axis \( L \) inside \( \mathbb{R}^2 \). Then \( g|_L \) is a continuous, injective map from \( \mathbb{R} \) to \( \mathbb{R} \). In particular, it has to be monotonic and its image must be an open interval around 0. The same argument for the \( y \)-axis \( L' \) shows that \( g|_{L'} \) has image an open interval around 0. This implies that \( g(L) \) and \( g(L') \) intersect in more than one point, violating the injectivity of \( g \). We thus have a contradiction.

5. Let us take \( A = 0 \) and \( G(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2) \). These clearly satisfy the two requirements. We see that any solution to \( u'(t) = Au(t) + G(u(t)) \) satisfies \( u_i(t) = \frac{a_i}{1-a_i} \) where \( a_i \) is the \( i \)-th coordinate of the initial condition \( u_0 \). For any \( \epsilon > 0 \), taking \( u_0 = \frac{\epsilon}{2} e_1 \) which satisfies \( ||u_0|| < \epsilon \), we get \( u(t) = \frac{1}{2-\epsilon} e_1 \Rightarrow ||u(t)|| = \frac{1}{2-\epsilon} \to +\infty \) as \( t \to \frac{2}{\epsilon} \).

6. (a) This is algebraic manipulation using the formulas \( x(t) = r(t) \cos \theta(t) \) and \( y(t) = r(t) \sin \theta(t) \). For example, dropping the \( t \) from the notation for brevity, we have \( r^2 = x^2 + y^2 \) and hence

\[
r' = x'x + y'y = x^2 - r(x^2 + xy) + y^2 + r(xy - y^2) - x^2y = r^2(1-r)
\]

Dividing by \( r \), we get \( r'(t) = r(t)(1-r(t)) \). This covers the case \( r(t) = 0 \) for some \( t \) too, since the origin is an equilibrium point for the system. We can derive the other formula analogously.

(b) In polar coordinates we need to show equivalently that \( r(t) \to 1 \) and \( \theta(t) \to 2n\pi \) as \( t \to +\infty \) for some integer \( n \).

We can solve the first equation for \( r(t) \) by using separation of variables to obtain \( \frac{r(t)}{r(t)-1} = \frac{r(0)}{r(0)-1} e^t \), which implies that \( r(t) \to 1 \) as \( t \to +\infty \) assuming \( r(0) \neq 0 \).

If \( R(t) \) is an anti-derivative for \( r(t) \), then we can use separation of variables
again together with the double-angle formula \(1 - \cos \theta = 2 \sin^2 \left( \frac{\theta}{2} \right)\) to solve the second equation and get \(\cot \left( \frac{\theta(t)}{2} \right) = -R(t) + C\) for some constant \(C\).

It is clear that since \(r(t) \to 1, R(t) \to +\infty\) as \(t \to +\infty\), which implies that \(\theta(t) \to 2n\pi\) as desired.

(c) Note that if the initial condition is \((r_0, \theta_0)\), where for example we can take \(r_0 < 1, \theta_0 > 0\) both close to 1 and 0 respectively, then since both \(r(t), \theta(t)\) are increasing by the equations of our system, we will have

\[
\theta'(t) = r(t)(1 - \cos \theta(t)) \geq r_0(1 - \cos \theta_0) > 0
\]

as long as \(\theta(t) < \pi\) for example. This clearly implies that for some time \(t_0\) we will have \(\theta(t_0) = \frac{\pi}{2}\) and hence the solution will move “far” from \((1,0)\). Therefore \((1,0)\) is not a stable equilibrium point.