1. It is clear that \( L(x) \) attains a strict global minimum at 0. We have 
\[ \nabla L(x) = 2x \] 
and hence 
\[ \langle \nabla L(x), Ax + G(x) \rangle = 2\langle x, Ax \rangle + 2\langle x, G(x) \rangle \]

Now 
\[ \frac{d}{dt} \| e^{At}u_0 \|^2 = \frac{d}{dt} \langle e^{At}u_0, e^{At}u_0 \rangle = 2\langle e^{At}u_0, Ae^{At}u_0 \rangle < -\alpha \| e^{At}u_0 \|^2 \]

So for \( t = 0 \) we obtain 
\[ 2\langle u_0, Au_0 \rangle < -\alpha \| u_0 \|^2 \]

Let \( \delta > 0 \) be such that 
\[ \| G(x) \| \leq \frac{\alpha}{2} \| x \| \Rightarrow 2\langle x, G(x) \rangle \leq \alpha \| x \|^2 \]

for all \( \| x \| < \delta \). For all such \( x \), it follows that 
\[ \langle \nabla L(x), Ax + G(x) \rangle < -\alpha \| x \|^2 + \alpha \| x \|^2 = 0 \]

This shows that \( L(x) \) is a Lyapunov function for 0 for the system 
\[ u'(t) = F(u(t)) = Au(t) + G(u(t)) \] 
and thus it is a stable equilibrium point.

2. (a) We have 
\[ u'(t) = Au(t) + G(u(t)) \Rightarrow \frac{d}{dt} (e^{-At}u(t)) = e^{-At}G(u(t)) \Rightarrow e^{-At}u(t) - u_0 = \int_0^t e^{-As}G(u(s))ds \Rightarrow u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}G(u(s))ds \]

It follows that 
\[ \| u(t) \| \leq \| e^{At} \|_{op} \| u_0 \| + e \int_0^t \| e^{A(t-s)} \|_{op} \| u(s) \| ds \leq \]
\[ \leq \Lambda e^{-\alpha t} \| u_0 \| + 2\Lambda^2 \epsilon \| u_0 \| \int_0^t e^{\alpha(s-t)}e^{-\alpha s/2}ds = \]
\[ = \Lambda e^{-\alpha t} \| u_0 \| + \frac{4\epsilon \Lambda^2}{\alpha} \| u_0 \| (e^{-\alpha t/2} - e^{-\alpha t}) \leq \]
\[ \leq \Lambda e^{-\alpha t} \| u_0 \| + \frac{8\epsilon \Lambda^2}{\alpha} e^{-\alpha t/2} \| u_0 \| \]

(b) Let \( s_* \) be as in the statement. Then we must have by continuity and 
the definition of \( s_* \) that 
\[ \| u(s_*) \| = 2\Lambda \| u_0 \| e^{-\alpha s_*/2} \]

By part (a), we deduce that 
\[ 2\Lambda \| u_0 \| e^{-\alpha s_*/2} \leq \Lambda e^{-\alpha s_*} \| u_0 \| + \frac{8\epsilon \Lambda^2}{\alpha} e^{-\alpha s_*/2} \| u_0 \| \Rightarrow \frac{\alpha}{8\Lambda} < \frac{\alpha}{4\Lambda} (1 - \frac{1}{2} e^{-\alpha s_*/2}) \leq \epsilon \]
For $\epsilon \leq \frac{\alpha}{8\Lambda}$, this is clearly a contradiction.

(c) By part (b), it readily follows that (2) holds true for all $s \geq 0$ and thus $\|u(s)\| \to 0$ exponentially as $s \to +\infty$ for all $\|u_0\| < r$, whence 0 is asymptotically stable.

3. (a) We have seen (or can easily compute) that $e^{At} = \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right)$. This is a rotation matrix and hence $\|e^{At}u\| = \|u\|$ for all $u$ and hence $\|e^{At}\|_{op} = 1$ for all $t$.

(b) We have

$$u'(t) = Au(t) + G(u(t)) \Rightarrow \frac{d}{dt} (e^{-At}u(t)) = e^{-At}G(u(t)) \Rightarrow e^{-At}u(t) - u_0 = \int_0^t e^{-A\tau}G(u(\tau)) d\tau \Rightarrow u(t) = e^{At}u_0 + \int_0^t e^{A(t-\tau)}G(u(\tau)) d\tau$$

It follows that, using $\|e^{At}\|_{op} = 1$ and $\|G(u)\| \leq \|u\|^p$

$$\|u(t)\| \leq \|u_0\| + \int_0^t \|u(\tau)\|^p d\tau \leq \|u_0\| + t \sup_{\tau \in [0,t]} \|u(\tau)\|^p$$

(c) Suppose not and consider $t_*$ as in the statement. Then by continuity we must have $\|u(t_*)\| = 2\epsilon$. Therefore, by part (b), we get

$$2\epsilon \leq \|u_0\| + t_* \sup_{\tau \in [0,t_*]} \|u(\tau)\|^p < \epsilon + 4^{-p}\epsilon^{p+1}2^p \epsilon^p \leq \epsilon + \frac{\epsilon}{4}$$

which is a contradiction.

(d) Using part (c) and applying the Extension Theorem, it is immediate that $(0, 4^{-p}\epsilon^{p+1})$ is contained in the maximal interval of existence.

4. Since the eigenvalues of $A$ have positive real part, we know that there exists $r > 0$ such that all solutions to $u' = Au + G(u)$ with $0 < \|u(0)\| < r$ must leave the ball $B_r(0)$ in finite time. Let $t_0$ be such that $\|u(t)\| < r$ for all $t \geq t_0$. Then $v(t) = u(t+t_0)$ is a solution of the given ODE that satisfies $\|v(t)\| < r$ for all $t \geq 0$. This implies that necessarily $v(0) = u(t_0) = 0$ and therefore $u(t) = 0$ for all $t \geq 0$.

5. (a) We have $\sin x = \cos y = 0 \Leftrightarrow (x, y) = (m\pi, n\pi + \pi/2)$, where $m, n \in \mathbb{Z}$. These are all the equilibrium points.

The linearization of the system at $(m\pi, n\pi + \pi/2)$ is given by

$$x'(t) = \cos(m\pi)x(t) = (-1)^mx(t), \quad y'(t) = -\sin(n\pi + \pi/2)y(t) = (-1)^ny(t)$$
We see that all equilibria are hyperbolic. If at least one of $m, n$ is even, the equilibrium is unstable, and otherwise, when both $m, n$ are odd, the equilibrium is asymptotically stable.

(b) The only equilibrium point is the origin. Let us set $x(t) = -y(t)$. The system becomes then $y'(t) = y(t)^2 \Rightarrow y(t) = \frac{y(0)}{1 - y(0)t}$. We see that for any initial condition $y(0) \neq 0$ we have $|y(t)| \to +\infty$ as $t \to \frac{1}{y(0)}$. Hence 0 is an unstable equilibrium point.