For problem 1, consider the ODE
\[ u'(t) = F(u(t)), \tag{1} \]
where \( F : \mathbb{R}^n \to \mathbb{R}^n \) is smooth.

1. (10 points) Prove that a bounded \( \omega \)-limit set is connected, i.e., show that if \( \Omega \) is a bounded \( \omega \)-limit set (which is therefore closed by what we have discussed), it cannot be written as \( \Omega = \Omega_1 \cup \Omega_2 \), where \( \Omega_1 \) and \( \Omega_2 \) are disjoint closed sets. [Hint: Suppose we have such \( \Omega_1 \) and \( \Omega_2 \), first show that they must be separated by some positive distance \( \delta > 0 \). Then show by the intermediate value theorem that there must be a point in \( \Omega \) which is distance \( \frac{\delta}{2} \) away from \( \Omega_1 \). Conclude that this is a contradiction.]

2. (30 points) Investigate further the following example from lectures: Consider the ODE
\[ x'(t) = \sin x(t)\left(-\frac{1}{10}\cos x(t) - \cos y(t)\right), \]
\[ y'(t) = \sin y(t)\left(\cos x(t) - \frac{1}{10}\cos y(t)\right). \]

(a) Show that \( \left(\frac{\pi}{2}, \frac{\pi}{2}\right), (0,0), (0, \pi), (\pi,0) \) and \( (\pi, \pi) \) are all unstable equilibrium points.

(b) Consider initial data \( (x(0), y(0)) \in (0, \pi) \times (0, \pi) \setminus \{(\frac{\pi}{2}, \frac{\pi}{2})\} \). Show that the \( \omega \)-limit set \( \Omega \subset Q \), where \( Q \) is defined to be the square bounded by \( x = 0, \pi \) and \( y = 0, \pi \). [Hint: Consider the function \( \sin x(t)\sin y(t) \).]

(c) Take the assumptions and notations as above. Show that \( \Omega \neq \{e\} \) for some equilibrium point \( e \). [Hint: Show that this would contradict the stable manifold theorem.]

(d) Take the assumptions and notations as above. Show that \( \Omega \) must contain at least one closed line segment of \( Q \). [Hint: Assume not. Use problem 1 and other properties of \( \omega \)-limit sets that we proved to show that \( \Omega = \{e\} \) for some equilibrium point \( e \).]

(e) (Finally, you do not have to turn this in, but you may want to think about why in fact \( \Omega = Q \).)