NOTES ON LINEAR ODES

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We can now use all the discussions we had on linear algebra to study linear ODEs. Most of this material appears in the textbook in §21, 22, 23, 26. As always, this is a preliminary version: if you have any comments or corrections, please send them to me.

1. General solutions to linear systems

Given a linear ODE

\[
\begin{align*}
\frac{du}{dt} &= Au(t) \\
 u(0) &= u_0,
\end{align*}
\]

we know that the unique solution is defined for all \( t \in \mathbb{R} \) and is given by

\[ u(t) = e^{At}u_0. \]

Given our discussions on taking exponentials of matrices, we immediately have

**Theorem 1.1.** Let \( A \) be an \((n \times n)\) complex matrix. The \( \ell \)-th component of any solution to \( \frac{du}{dt} = Au(t) \) takes the following form

\[ u_\ell(t) = \sum_{\lambda_j \text{ eigenvalue of } A} p_{\ell,j}(t)e^{\lambda_j t}, \]

where for each \( \ell, j \), \( p_{\ell,j}(t) \) is a polynomial.

**Proof.** Recall that every matrix is similar to a matrix in the Jordan canonical form, i.e., there exists \( S \) such that

\[ S^{-1}AS = J, \]

where \( J \) is in the Jordan canonical form. Moreover, as we discussed earlier, every non-zero entry of \( e^{Jt} \) takes the form \( e^{\lambda_i t} \frac{t^k}{k!} \) (for some eigenvalue \( \lambda_i \) and some \( k \in \mathbb{N} \)). This then implies the theorem. \( \square \)

**Remark 1.2.** Notice that not all functions of the above form are solutions.

**Remark 1.3.** Suppose the characteristic polynomial is given by \( \chi_A(\lambda) = \prod_{i=1}^m (\lambda - \lambda_i)^{\nu_i} \), where \( \lambda_i \) are distinct. Then the polynomials \( p_{\ell,j}(t) \) have degree strictly smaller than \( \nu_j \) (Exercise: Why?)

If \( A \) is real, we want to find an expression for general real solutions. These are in particular the solutions one obtain when given real initial data. Before we proceed, let us just make a simple observation that if \( A \) is an \((n \times n)\) real matrix and \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) is an eigenvalue, then its complex conjugate \( \bar{\lambda} \) is also an eigenvalue. (This is because if \( Av = \lambda v \) for \( v \in \mathbb{C}^n \setminus \{0\} \), then \( A\bar{v} = \bar{\lambda}\bar{v} \).) With this observation, we have the following corollary:

**Corollary 1.4.** Let \( A \) be an \((n \times n)\) real matrix. Suppose \( \mu_1, \ldots, \mu_m \) are all of its real eigenvalues and \( \alpha_1 \pm i\beta_1, \ldots, \alpha_s \pm i\beta_s \) are all of its eigenvalues in \( \mathbb{C} \setminus \mathbb{R} \). Then the \( \ell \)-th component of any real solution to \( \frac{du}{dt} = Au(t) \) takes the following form

\[
\begin{align*}
u_\ell(t) = \sum_{i=1}^m p_{\ell,i}(t)e^{\mu_i t} + \sum_{k=1}^s e^{\alpha_k t}(q_{\ell,k}(t)\sin(\beta_k t) + r_{\ell,k}(t)\cos(\beta_k t)),
\end{align*}
\]

where \( p_{\ell,i}(t), q_{\ell,k}(t) \) and \( r_{\ell,k}(t) \) are real polynomials.
Proposition 2.2. Let \( A \) be an \((n \times n)\) complex matrix such that all eigenvalues \( \lambda_j \) satisfy \( \text{Re}(\lambda_j) < -\alpha < 0 \). Then, for any solution \( u(t) \) to \( u'(t) = Au(t) \), we have
\[
e^{\alpha t} \|u(t)\| \to 0
\]
as \( t \to \infty \).

Proof. By Theorem 1.1, each component \( u_\ell(t) \) takes the form
\[
u_\ell(t) = \sum_{\lambda_j \text{ eigenvalue of } A} p_{\ell,j}(t)e^{\lambda_j t}.
\]
Since \( \text{Re}(\lambda_j) < -\alpha < 0 \), \( e^{\alpha t} \times \|(\text{RHS})\| \to 0 \) for each \( \ell \) as \( t \to \infty \). The proposition hence follows.

Although Proposition 2.1 above shows that all solutions converge to 0 as \( t \to \infty \), it does not actually show that 0 is a stable equilibrium. (This is because Proposition 2.1 is in principle consistent with solutions becoming very large before converging to 0). Nevertheless, we have the following more quantitative proposition\footnote{Note that strictly speaking this proposition only give the boundedness of \( e^{\alpha t} \|u(t)\| \) instead of its decay. Nevertheless, since the inequality \( \text{Re}(\lambda_j) < -\alpha < 0 \) is strict, there exists \( \beta > 0 \) such that \( \text{Re}(\lambda_j) < -\beta < -\alpha < 0 \). Hence, by applying Proposition 2.2 with \( \beta \) in place of \( \alpha \), we see that its conclusion indeed implies Proposition 2.1.}:

Proposition 2.2. Let \( A \) be an \((n \times n)\) complex matrix such that all eigenvalues \( \lambda_j \) satisfy \( \text{Re}(\lambda_j) < -\alpha < 0 \). Then there exists \( \Lambda > 0 \) such that for every solution \( u(t) \) to \( u'(t) = Au(t) \),
\[
\|u(t)\| \leq \Lambda e^{-\alpha t}\|u(0)\|
\]
for all \( t \geq 0 \).

Proof. Since \( u(t) = e^{At}u(0) \), it suffices to prove that \( \|e^{At}\|_{op} \leq \Lambda e^{-\alpha t} \). By our discussions on linear algebra, there exists an invertible matrix \( S \) such that \( S^{-1}AS = J \), where \( J \) is in the Jordan canonical form. Hence, \( e^{At} = Se^{Jt}S^{-1} \). Recall that every non-zero entry of \( e^{Jt} \) takes the form \( e^{\lambda_k t} \frac{k!}{k!} \) (for some eigenvalue \( \lambda_k \) and some \( k \in \mathbb{N} \)). By Problem 1 in HW3, we know that \( \|e^{Jt}\|_{op} \leq (\sum_{ij}|(e^{Jt})_{ij}|^2)^{\frac{1}{2}} \) (where \( (e^{Jt})_{ij} \) denotes the \( ij \)-th entry of \( e^{Jt} \)). Therefore, there exists a constant \( C > 0 \) such that
\[
\|e^{At}\|_{op} \leq C \sum_{\lambda_j \text{ eigenvalue of } A} e^{(\alpha+\text{Re}(\lambda_j))t}(1 + t + \cdots + t^{n-1}),
\]
which is bounded for all \( t \geq 0 \) since \( \text{Re}(\lambda_j) < -\alpha \). Finally, since
\[
\|e^{At}\|_{op} \leq \|S\|_{op}\|e^{Jt}\|_{op}\|S^{-1}\|_{op},
\]
we obtain the conclusion of the proposition. \( \square \)

Next, we consider the case where some eigenvalues can have negative real parts.
Theorem 2.3. Suppose $A$ is a $(n \times n)$ complex matrix. Assume that there are no eigenvalues on the imaginary axis and that every eigenvalue satisfies $|Re(\lambda_i)| > \alpha$ for some $\alpha > 0$. Then $\mathbb{C}^n$ decomposes as
$$C^n = P_+ \oplus P_-,$$
so that
$$e^{at}\|e^{At}u_0\| \to 0, \quad \text{as } t \to \infty, \text{ for } u_0 \in P_- \tag{2.1}$$
and
$$e^{-at}\|e^{At}u_0\| \to \infty, \quad \text{as } t \to \infty, \text{ for } u_0 \in P_+ \setminus \{0\}. \tag{2.2}$$

Proof. Definitions of $P_+$ and $P_-$. Recall that $\mathbb{C}^n$ can be decomposed as
$$\mathbb{C}^n = \ker((\lambda_1 I - A)^{\nu_1}) \oplus \cdots \oplus \ker((\lambda_k I - A)^{\nu_k}) \oplus \ker((\lambda_{k+1} I - A)^{\nu_{k+1}}) \oplus \cdots \oplus \ker((\lambda_m I - A)^{\nu_m}).$$

Without loss of generality (by otherwise relabelling), assume that
$$Re(\lambda_1), \ldots, Re(\lambda_k) > \alpha > 0$$
and
$$Re(\lambda_{k+1}), \ldots, Re(\lambda_n) < -\alpha < 0.$$

Define
$$P_+ = \ker((\lambda_1 I - A)^{\nu_1}) \oplus \cdots \oplus \ker((\lambda_k I - A)^{\nu_k})$$
and
$$P_- = \ker((\lambda_{k+1} I - A)^{\nu_{k+1}}) \oplus \cdots \oplus \ker((\lambda_m I - A)^{\nu_m}).$$

Clearly, $\mathbb{C}^n = P_+ \oplus P_-.$

**Property of $P_-$.** Let us first consider the case where $v_i \in \ker((\lambda_i I - A)^{\nu_i})$ for some $i = k + 1, \ldots, m$. Recall that $A = B + N$ where $B$ is diagonalizable, $N^n = 0$ and $BN = NB$. Moreover, we showed that $Bv_i = \lambda_i v_i$. Hence, for some $C > 0$, we have
$$e^{at}\|e^{At}v_i\| = e^{at}\|e^{Bt}e^{Nt}v_i\| = e^{at}\|e^{\lambda_i t}e^{Nt}v_i\| = e^{a(\alpha + Re(\lambda_i))t}\|e^{Nt}v_i\| \leq Ce^{(a + Re(\lambda_i))t}(1 + t + \cdots + t^{n-1}) \to 0.$$ 
Since any $u_0 \in P_-$ can be written as a sum of such $v_i$’s, the desired estimate for $e^{at}\|e^{At}u_0\|$ follows from the above discussion and the triangle inequality.

**Property of $P_+$, Step 1: A reduction.** We claim that to estimate $\|e^{At}u_0\|$ for $u_0 \in P_+$, it suffices to consider the case where $u_0 \in \ker((\lambda_i I - A)^{\nu_i}) \setminus \{0\}$ for some $i = 1, \ldots, k$. To see this, first notice that if $u_0 \in P_+$, then $e^{At}u_0 \in P_+$ for all $t \geq 0$. Moreover, we can define a norm (Exercise: Check that it is a norm) $\| \cdot \|_*$ on $P_+$ by
$$\|v\|_* = \sum_{i=1}^k \|v_i\|,$$
where
$$v = v_1 + \cdots + v_k, \quad v_i \in \ker((\lambda_i I - A)^{\nu_i}).$$

Since $P_+$ is finite dimensional, $\| \cdot \|_*$ must be equivalent to $\| \cdot \|$ on $P_+$, i.e., there exists $C > 0$ such that for every $v \in P_+$,
$$C^{-1}\|v\| \leq \|v\|_* \leq C\|v\|.$$ 

Now suppose we can show that $e^{-at}\|e^{At}v_i\| \to \infty$ whenever $v_i \in \ker((\lambda_i I - A)^{\nu_i}) \setminus \{0\}$ for some $i = 1, \ldots, k$. Given $u_0 \in P_+ \setminus \{0\}$, $u_0 = v_1 + \cdots + v_k$, $v_i \in \ker((\lambda_i I - A)^{\nu_i})$. At least one of these $v_i$’s is nonzero. Hence, $e^{-at}\|e^{At}u_0\| \geq C^{-1}e^{-at}\|e^{At}v_i\|_* \geq C^{-1}\sup_i e^{-at}\|e^{At}v_i\| \to \infty.$

**Property of $P_+$, Step 2.** Now let $u_0 \in \ker((\lambda_i I - A)^{\nu_i}) \setminus \{0\}$ for some $i = 1, \ldots, k$. We have
$$\|e^{At}u_0\| = \|e^{Bt}e^{Nt}u_0\| = \|e^{\lambda_i t}e^{Nt}u_0\| = e^{\alpha t + Re(\lambda_i)t}\|e^{Nt}u_0\|. \tag{2.3}$$

We claim that $\|e^{Nt}u_0\|$ is bounded below for large $t$. To see this, note that since $N^n = 0$, $e^{Nt} = \sum_{i=0}^{n-1} \frac{t^i N^i}{i!}$. Hence, there exists a largest $i_0 \geq 0$ such that $N^{i_0}u_0 \neq 0$. Moreover, since all the other non-zero terms have smaller powers of $t$, for $t$ sufficiently large, $\|e^{Nt}u_0\| \geq \|e^{\lambda_{i_0} t}e^{N_{i_0}t}u_0\| =: \gamma > 0.$ Hence, by (2.3), we have that if $u_0 \in \ker((\lambda_i I - A)^{\nu_i}) \setminus \{0\}$ for some $i = 1, \ldots, k$, 
$$e^{-at}\|e^{At}u_0\| = e^{-at+Re(\lambda_i)t}\|e^{Nt}u_0\| \geq \gamma e^{-at+Re(\lambda_i)t} \to \infty.$$
Together with part 1, this concludes the proof. \qed
Remark 2.4. Notice that (2.1) defines $P_-$ uniquely, but (2.2) does not define $P_+$ uniquely. We will nevertheless from now on use $P_\pm$ to mean the subspaces constructed in Theorem 2.3.

Remark 2.5. $P_-$ is called the stable subspace (or stable manifold, or incoming manifold) and $P_+$ is called the unstable subspace (or unstable manifold, or outgoing manifold).

Remark 2.6. As long as $P_+ \neq \{0\}$, 0 is an unstable equilibrium. Moreover, for $u_0 = u_+ + u_-$ with $u_\pm \in P_\pm$, $\|e^{At}u_0\| \to \infty$ as long as $u_+ \neq 0$. In particular, there exists an open and dense set $U \subseteq \mathbb{C}^n$ such that $u_0 \in U$ implies $\|e^{At}u_0\| \to \infty$.

Remark 2.7. The discussion above assumes that $A$ is a complex matrix. In the case where $A$ is real, in fact one can also decompose

$$\mathbb{R}^n = \mathbb{R}P_+ \oplus \mathbb{R}P_-,$$

where $\mathbb{R}P_\pm = P_\pm \cap \mathbb{R}^n$.

The above discussion does not cover the case where at least one eigenvalue lies on the imaginary axis. That case is in general more complicated and we will not formulate a general theorem. In the following two examples, we will see that even if 0 is the only eigenvalue, there can be rather different long time behavior. In particular, in Example 2.8, 0 is a stable equilibrium while in Example 2.9, 0 is not a stable equilibrium.

Example 2.8. Let $A$ be the $(2 \times 2)$ zero matrix. In this case, any solution to $u'(t) = Au(t) = 0$ is constant in time. In particular, it neither grows nor decays as $t \to \infty$.

Example 2.9. In the second example, consider

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$ 

The eigenvalues are also 0 as in the previous example. Nevertheless,

$$e^{At}u_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (u_0)_1 \\ (u_0)_2 \end{bmatrix} = \begin{bmatrix} (u_0)_1 + t(u_0)_2 \\ (u_0)_2 \end{bmatrix}.$$ 

Hence, the solution grows as $t \to \infty$ if $(u_0)_2 \neq 0$. Finally, let us note that in both Proposition 2.1 and Example 2.8, 0 is a stable equilibrium, but they have somewhat different long time behavior in that $u(t) \to 0$ in Proposition 2.1 but $u(t) \not\to 0$ in general in Example 2.8. It is therefore useful to make the following definition:

**Definition 2.10 (Asymptotic stability).** We say that a solution $\bar{u}(t)$ to an ODE with initial data $\bar{u}(0) = \bar{u}_0$, which is defined for all $t \geq 0$, is asymptotically stable if both of the following hold:

- $\bar{u}$ is stable, and
- there exists $r > 0$ such that for any data $u_0 \in B(\bar{u}_0, r)$, the corresponding solution $u(t)$ satisfies $\|u(t) - \bar{u}(t)\| \to 0$ as $t \to \infty$.

**Remark 2.11.** Let us note that the assumption that $\bar{u}$ is stable is necessary. (Exercise: Why?)

### 3. Equivalence of linear systems

**Definition 3.1.** Let $A$ and $B$ be two $(n \times n)$ real matrices. Two linear ODEs

$$u'(t) = Au(t)$$

and

$$u'(t) = Bu(t)$$

are said to be

- linearly equivalent as real ODEs if there exists an invertible function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

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- $\bar{u}$ is stable, and
- there exists $r > 0$ such that for any data $u_0 \in B(\bar{u}_0, r)$, the corresponding solution $u(t)$ satisfies $\|u(t) - \bar{u}(t)\| \to 0$ as $t \to \infty$.

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### 3. Equivalence of linear systems

**Definition 3.1.** Let $A$ and $B$ be two $(n \times n)$ real matrices. Two linear ODEs

$$u'(t) = Au(t)$$

and

$$u'(t) = Bu(t)$$

are said to be

- linearly equivalent as real ODEs if there exists an invertible function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that
Proof. It is clear that if there is an invertible matrix $A$, we have

$$f \circ e^{At} = e^{Bt} \circ f$$

as maps $\mathbb{R}^n \to \mathbb{R}^n$ for all $t \in \mathbb{R}$.

Remark 3.2. It is easy to check (Exercise) that these are indeed equivalence relations.

Remark 3.3. It is clear from the definition that

linear equivalence $\implies$ smooth equivalence $\implies$ topological equivalence.

Remark 3.4. One can also entertain the various notions of equivalence for complex ODEs. We will however not discuss that in these notes.

We now study each of these notions of equivalences. First, we start with linear equivalence.

Proposition 3.5. Let $A$ and $B$ be two real $(n \times n)$ matrices. The ODEs $u'(t) = Au(t)$ and $u'(t) = Bu(t)$ are linearly equivalent as real ODEs if and only if $A$ and $B$ are similar.

Proof. It is clear that if there is an invertible matrix $C$ such that $C^{-1}AC = B$, then $Ce^{At} = e^{Bt}C$.

Conversely, if there is an invertible matrix $C$ such that $Ce^{At} = e^{Bt}C$, by differenting and evaluating at $t = 0$, we obtain $AC = BC$. \[\square\]

We then turn to smooth equivalence. It turns out that it is the same as linear equivalence.

Theorem 3.6. Two linear real ODEs are linearly equivalent (as real ODEs) if and only if they are smoothly equivalent.

Proof. As we noted in Remark 3.3, linear equivalence $\implies$ smooth equivalence is easy. We now prove the other implication.

Let $A$ and $B$ be two real $(n \times n)$ matrices. Suppose there is a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f^{-1}$ well-defined and smooth such that

$$f \circ e^{At} = e^{Bt} \circ f.$$

Step 1. We first note that since $e^{At}0 = 0$, we must have that $e^{Bt}f(0) = f(0)$. Define

$$\tilde{f}(x) = f(x) - f(0).$$

Clearly $\tilde{f}$ is smooth, invertible, and $\tilde{f}^{-1}$ is smooth. We check that

$$\tilde{f}(e^{At}x) = f(e^{At}x) - f(0) = e^{Bt}f(x) - f(0) = e^{Bt}(f(x) - f(0)) = e^{Bt}\tilde{f}(x). \tag{3.1}$$

Step 2. Since $\tilde{f}$ is differentiable and $\tilde{f}(0) = 0$, we have

$$\tilde{f}(x) = (D\tilde{f}_0)x + G(x),$$

where $(D\tilde{f}_0)$ is a (constant) $(n \times n)$ invertible matrix, and $G$ satisfies $\lim_{\|x\| \to 0} \frac{\|G(x)\|}{\|x\|} = 0$. Now, using (3.1), we have

$$(D\tilde{f}_0)e^{At}x - e^{Bt}(D\tilde{f}_0)x = -G(e^{At}x) + e^{Bt}G(x).$$

In particular, given fixed $t \in \mathbb{R}$, $x \in \mathbb{R}^n \setminus \{0\}$, for every $\epsilon > 0$, there exists $\delta > 0$ sufficiently small such that

$$\|\delta(D\tilde{f}_0)e^{At}x - e^{Bt}(D\tilde{f}_0)x\| \leq \| - G(\delta e^{At}x) + e^{Bt}G(\delta x)\| \leq C\delta \|x\|.$$

for some $C > 0$. This implies

$$\|(D\tilde{f}_0)e^{At}x - e^{Bt}(D\tilde{f}_0)x\| \leq C\epsilon \|x\|.$$

Since $\epsilon$ is arbitrary, we have

$$(D\tilde{f}_0)e^{At} = e^{Bt}(D\tilde{f}_0).$$

The conclusion follows. \[\square\]
Remark 3.7. This does not mean if there is a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f^{-1}$ well-defined and smooth satisfying
\[ f(e^{At}u_0) = e^{Bt}f(u_0), \]
then $f$ is linear. (Exercise: Why?)

Finally, we discuss topological equivalence. The main result is in Theorem 3.17. We will first start with some preliminary discussions.

**Proposition 3.8.** Let $A$ be an $(n \times n)$ complex matrix. Let $\epsilon > 0$, then there exist an invertible matrix $S$ and an upper triangular matrix $T$ such that
\[ S^{-1}AS = T, \]
\[ \text{All the entries of } T \text{ above the diagonal have size } < \epsilon. \]

**Proof.** Since we know that every matrix is similar to one in the Jordan canonical form, it suffices to prove the proposition for a Jordan block
\[
\lambda I + J_n := \begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \lambda \\
0 & \cdots & \cdots & 0 & \lambda
\end{bmatrix}.
\]

Define $S$ to be the diagonal matrix
\[
S = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \epsilon & 0 & \cdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \epsilon^{-1}
\end{bmatrix}.
\]

Then
\[
S^{-1}(\lambda I + J_n)S = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \epsilon^{-1} & 0 & \cdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \epsilon^{-1}
\end{bmatrix} \begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \lambda \\
0 & \cdots & \cdots & 0 & \lambda
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \epsilon & 0 & \cdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \epsilon^{-1}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \epsilon & 0 & \cdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \epsilon^{-1}
\end{bmatrix} \begin{bmatrix}
\lambda & \epsilon & 0 & \cdots & 0 \\
0 & \epsilon & 0 & \cdots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \lambda \\
0 & \cdots & \cdots & 0 & \epsilon
\end{bmatrix}.
\]
\[ \square \]
Therefore, by the implicit function theorem

\[ \begin{array}{c}
\text{Proposition 3.9. Let } A \text{ be a complex } (n \times n) \text{ matrix whose eigenvalues have positive real parts. Then there exists a norm } \| \cdot \|_* \text{ on } \mathbb{C}^n \text{ such that } \frac{d}{dt} \| e^{At}u_0 \|_*^2 > 0 \text{ for all } u_0 \in \mathbb{C}^n \setminus \{0\}.
\end{array} \]

**Proof.** Let \( \epsilon > 0 \) (to be chosen later). By Proposition 3.8, there exists \( S \) such that \( S^{-1}AS = T \) for some upper triangular \( T \) with off-diagonal entries < \( \epsilon \).

Define

\[ \|z\|^2 := \sum_{i=1}^n (S^{-1}z)_i(S^{-1}z)_i. \]

It is easy to check that \( \| \cdot \|_* \) is a norm.

We compute

\[ \frac{d}{dt} \| e^{At}z \|^2 = \frac{d}{dt} \| Se^{Tt}S^{-1}z \|^2 = \sum_{i=1}^n 2Re((e^{Tt}S^{-1}z)_i(e^{Tt}TS^{-1}z)_i) \geq 2 \min_\lambda \| e^{Tt}S^{-1}z \|^2 - C \| e^{Tt}S^{-1}z \|^2, \]

for some \( C > 0 \) depending only on \( n \), where \( \| \cdot \| \) is the standard norm on \( \mathbb{C}^n \) given by \( \|z\|^2 = \sum_{i=1}^n |z_i|^2 \). By our assumptions on the eigenvalues of \( A \), we can choose \( \epsilon \) sufficiently small so that the RHS is > 0. \( \square \)

**Theorem 3.10.** Let \( A \) be a real \((n \times n)\) matrix so that all eigenvalues have positive real parts. Then

\[ u'(t) = u(t), \quad u'(t) = Au(t) \]

are topologically equivalent as real ODEs.

**Proof.** Let \( \| \cdot \|_* \) be the norm as in Proposition 3.9. Define the homeomorphism by

\[ f(x) = \begin{cases} e^{A \log \|x\|} \frac{x}{\|x\|_*} & x \neq 0, \\ 0 & x = 0. \end{cases} \]

We check that it has the desired properties:

\( f \) gives the equivalence of the ODEs. To see this, we check that

\[ f(e^{\epsilon}x) = e^{A \log \|e^{\epsilon}x\|} \frac{e^{\epsilon}x}{\|e^{\epsilon}x\|_*} = e^{At} e^{A \log \|x\|} \frac{x}{\|x\|_*} = e^{At} f(x). \]

\( f \) is continuous. This is obvious when \( x \neq 0 \). To see that \( f \) is continuous at 0, we check that

\[ \lim_{\|x\|_* \to 0} e^{A \log \|x\|} \frac{x}{\|x\|_*} = 0, \]

since by Proposition 2.2, \( \lim\|x\|_* \to 0 \|e^{A \log \|x\|} \|_* \|_* = 0 \).

\( f^{-1} \) is well-defined. \( f^{-1} \) can be given by

\[ f^{-1}(y) = \begin{cases} e^{-At} e^{\tau} y, & y \neq 0, \\ 0 & y = 0 \end{cases} \]

for \( \tau \) such that

\[ \|e^{-At}y\|_* = 1. \]

We need to show that \( \tau \) exists and is unique for any given \( y \in \mathbb{R}^n \setminus \{0\} \). The existence of \( \tau \) follows from \( \lim_{\tau \to -\infty} \|e^{-At}y\|_* = +\infty \), \( \lim_{\tau \to \infty} \|e^{-At}y\|_* = 0 \) and the intermediate value theorem. The uniqueness of \( \tau \) follows from Proposition 3.9.

\( f^{-1} \) is continuous. We first check that \( f^{-1} \) is continuous away from 0. It suffices to show that the map \( y \mapsto \tau : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) is continuous. For this, notice that by Proposition 3.9, for \( y \neq 0 \),

\[ \frac{d}{d\tau} \|e^{-At}y\|^2 < 0. \]

Therefore, by the implicit function theorem\(^3\), \( \tau \) is \( C^1 \) and therefore continuous at \( y \).

\(^3\)Here, let us recall the following particular case of the implicit function theorem. Let \( g : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) function. Suppose there exists \((x_0, y_0) \in \mathbb{R}^k \times \mathbb{R} \) such that

\[ g(x_0, y_0) = 0 \]
Next, we check that \( f^{-1} \) is continuous at 0. Suppose there exists a sequence \( y_n \to 0 \). In order that 
\[
\|e^{A\tau_n}y_n\| = 1
\]
we have \( \tau_n \to -\infty \). Hence, \( \|f^{-1}(y_n)\| \leq e^{\tau_n} \to 0 \), as desired.
\[\square\]

Remark 3.11. Since topological equivalence is an equivalence relation, the above theorem in particular shows that if \( A \) and \( B \) are both real \((n \times n)\) matrices so that all eigenvalues have positive real parts. Then
\[
u'(t) = Bu(t), \quad u'(t) = Au(t)
\]
are topologically equivalent as real ODEs.

In a completely analogous manner, we also have

**Proposition 3.12.** Let \( A \) be a real \((n \times n)\) matrix so that all eigenvalues have negative real part. Then
\[
u'(t) = -u(t), \quad u'(t) = Au(t)
\]
are topologically equivalent as real ODEs.

Combining Theorem 3.10 and Proposition 3.12, we have

**Theorem 3.13.** Let \( A \) be a real \((n \times n)\) matrix so that exactly \( n^+ \) eigenvalues (counting multiplicities) have positive real part, exactly \( n^- \) eigenvalues (counting multiplicities) have negative real part, and no eigenvalues are on the imaginary axis. Let \( I_{n^+,n^-} \) be the diagonal matrix \( I_{n^+,n^-} = \text{diag}(+1,+1,\ldots,+1,-1,\ldots,-1) \), where the first \( n^+ \) diagonal entries are +1 and the last \( n^- \) diagonal entries are −1. Then
\[
u'(t) = I_{n^+,n^-}u(t), \quad u'(t) = Au(t)
\]
are topologically equivalent as real ODEs.

**Proof.**

**Step 1.** Since \( \mathbb{R}^n = \mathbb{R}^+ \oplus \mathbb{R}^- \), we can find \( S \) so that \( S^{-1}AS = \tilde{A} \), which takes the following block diagonal form
\[
\tilde{A} = \begin{bmatrix}
A^+ & 0 \\
0 & A^-
\end{bmatrix}
\]
for some \( n^+ \times n^+ \) real matrix \( A^+ \), whose eigenvalues have positive real parts, and some \( n^- \times n^- \) real matrix \( A^- \), whose eigenvalues have negative real parts.

**Step 2.** By Remark 3.3 and Proposition 3.5, it suffices to show that
\[
u'(t) = I_{m,n}u(t), \quad u'(t) = \tilde{A}u(t)
\]
are topologically equivalent as real ODEs, where \( \tilde{A} \) is as in Step 1. By Theorem 3.10 and Proposition 3.12, there exists \( f_\pm : \mathbb{R}^{n\pm} \to \mathbb{R}^{n\pm} \) which is continuous with continuous inverse such that
\[
f_\pm \circ e^{A^\pm t} = e^{A^\pm t} \circ f_\pm.
\]
Writing \( x \in \mathbb{R}^n \) as \( x = (x^+, x^-) \), where \( x^\pm \in \mathbb{R}^{n\pm} \), define
\[
f(x) = (f_+(x^+), f_-(x^-)).
\]
It is easy to see that \( f \) has the desired properties to imply the conclusion of the Theorem.
\[\square\]

In the above, we have given a sufficient condition for topological equivalence of linear ODEs. It turns out that if we restrict to linear ODEs so that the corresponding matrix has no eigenvalues on the imaginary axis, then this condition is in fact necessary (see Proposition 3.16). To see this, we first need a fact.

**Definition 3.14.** We say that two metric spaces \((X, d)\) and \((Y, d)\) are **homeomorphic** if there exists a continuous bijection \( f : X \to Y \) such that \( f^{-1} \) is also continuous.

**Fact.** \( \mathbb{R}^n \) and \( \mathbb{R}^m \) are homeomorphic if and only if \( m = n \).
Remark 3.15. This seemingly obvious fact is not so easy to prove. Notice for instance that there exists a continuous surjection $\mathbb{R}^1 \to \mathbb{R}^2$.

**Proposition 3.16.** Let $A$ and $B$ be $(n \times n)$ real matrices. Suppose there exists a continuous $f : \mathbb{R}^n \to \mathbb{R}^n$ with continuous inverse such that

$$f \circ e^{At} = e^{Bt} \circ f.$$

Then

**Proof.** Suppose there exists an $f$ as in the statement of the proposition. Notice that

$$\mathbb{R}P_-(A) = \{x \in \mathbb{R}^n : \|e^{At}x\| \to 0 \text{ as } t \to \infty\}$$

and

$$\mathbb{R}P_-(B) = \{x \in \mathbb{R}^n : \|e^{Bt}x\| \to 0 \text{ as } t \to \infty\}.$$

Therefore, $f$ must be a homeomorphism between $\mathbb{R}P_-(A)$ and $\mathbb{R}P_-(B)$. By the fact above, they must have the same dimensions. \qed

Combining Theorem 3.13 and Proposition 3.16, we have the following

**Theorem 3.17.** Let $A$ and $B$ be $(n \times n)$ matrices with no eigenvalues on the imaginary axis. Then $u'(t) = Au(t)$ and $u'(t) = Bu(t)$ are topologically equivalent as real ODEs if their stable subspaces have the same dimensions.