1. We have, since \( u(0) = 0 \),

\[
\frac{d}{dt}u(t) = \cos\left(\frac{2t}{\pi}\right) (1 + u(t)^2) \implies \frac{du}{1 + u^2} = \cos\left(\frac{2t}{\pi}\right) \implies
\]

\[
\int_0^t \frac{du(\tau)}{1 + u(\tau)^2} d\tau = \int_0^t \cos\left(\frac{2\tau}{\pi}\right) d\tau \implies
\]

\[
\arctan(u(t)) - \arctan(u(0)) = \frac{\pi}{2} \sin\left(\frac{2t}{\pi}\right) \implies
\]

\[
u(t) = \tan\left(\frac{\pi}{2} \sin\left(\frac{2t}{\pi}\right)\right)
\]

For \( u(t) \) to be defined, we need \(-\frac{\pi}{2} < \frac{\pi}{2} \sin\left(\frac{2t}{\pi}\right) < \frac{\pi}{2}\) and thus \(-\frac{\pi^2}{4} < t < \frac{\pi^2}{4}\).

The maximal interval of existence is \( J = \left(-\frac{\pi^2}{4}, \frac{\pi^2}{4}\right) \).

2. If \( \alpha \in (0, 1) \), then we may find using separation of variables or by checking directly that \( u(t) = ((1 - \alpha)t)^{\frac{1}{1-\alpha}} \) and \( u(t) \equiv 0 \) are both solutions to the equation.

Let \( F(x) = |x|^{\alpha} \). If \( \alpha \geq 1 \), then the function \( G(x) = x^{\alpha} \) is continuously differentiable for \( x \geq 0 \) with derivative \( G'(x) = \alpha x^{\alpha-1} \). Thus for \( |y|, |x| < r \) we have

\[
|F(x) - F(y)| = |G(|x|) - G(|y|)| = \left| \int_{|y|}^{|x|} \alpha s^{\alpha-1} ds \right| \leq \alpha \max\{|x|^\alpha-1, |y|^{\alpha-1}\} ||x| - |y|| \leq L|x - y|
\]

where \( L = \alpha r^{\alpha-1} \).

Therefore, \( F \) satisfies the Lipschitz condition of the Picard-Lindelöf theorem, and the solution to \( u' = F(u) \) with \( u(0) = 0 \) is unique and since \( u \equiv 0 \) is a solution, it has to be the only solution to the initial value problem.

3. Let \( F \) be the vector field in question.

We first find the Hamiltonian \( H(p, q) \) for the system. We have

\[
-q - q^3 = -\frac{\partial H}{\partial q} \implies H(p, q) = \frac{q^2}{2} + \frac{q^4}{4} + f(p)
\]

Therefore we obtain from the second relation that

\[
p = \frac{\partial H}{\partial p} \iff f'(p) = p \iff f(p) = \frac{p^2}{2} + C
\]
Hence we may take $H(p, q) = \frac{p^2}{2} + \frac{q^2}{2} + \frac{q^4}{4}$.

Since $H$ is a Hamiltonian, we have $\langle \nabla H, F \rangle = 0$ and moreover it is clear that $H$ has a strict local minimum at $(0, 0)$. It is thus a Lyapunov function and by Lyapunov’s theorem it follows that $(0, 0)$ is a stable equilibrium.

4. (a) False. The solution to $u'(t) = a(t)u(t)$ is given by $u(t) = e^{\int_0^t a(s)ds}u(0)$.

For $a(t) = \cos t$ with period $T = 2\pi$ and any $u(0) \neq 0$ we obtain $u(t) = e^{\sin t} u(0)$, which is always periodic of period $T = 2\pi$ and not identically zero.

(b) True. Let $\epsilon > 0$ be given. Then there exist:

- $\delta_\infty > 0$ such that $|f(t) - f(s)| < \frac{\epsilon}{3}$ whenever $|t - s| < \delta_\infty$, since $f$ being continuous on a closed interval implies that it is uniformly continuous.

- $N > 0$ such that $|f_n(t) - f(t)| < \frac{\epsilon}{3}$ for all $t$ and $n > N$.

- For each $i = 1, ..., N$, $\delta_i > 0$ such that $|f_n(t) - f_n(s)| < \epsilon$ for all $|t - s| < \delta_i$.

Let $\delta = \min\{\delta_1, ..., \delta_N, \delta_\infty\}$.

Let $|t - s| < \delta$. If $n \leq N$, we have $|f_n(t) - f_n(s)| < \epsilon$. If $n > N$, then

$$|f_n(t) - f_n(s)| \leq |f_n(t) - f(t)| + |f(t) - f(s)| + |f_n(s) - f(s)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

We conclude that $\{f_n\}$ is equicontinuous.

(c) True. This is Newton’s equation with $V(y) = \frac{y^4}{4}$. Then $V(y) \geq 0$ for all $y$ and by Theorem 2.5 in the notes for Hamiltonian systems it follows that the maximal interval of existence is $\mathbb{R}$.

(d) True. We know that $y(t) = \frac{1}{1+t}$ satisfies $y' = y^2$ with $y(0) = 1$ and its maximal interval of existence is $J = (-\infty, 1)$. But then

$$y'' = (y')' = (y^2)' = 2yy' = 2y^3 = -V'(y)$$

where we may take $V(y) = -\frac{y^4}{2}$. Therefore, for $V(y) = -\frac{y^4}{2}$ we have a solution with maximal interval of existence $(-\infty, 1)$. 

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