Spring 2017 Math 63CM: Midterm 1

Answer all questions. This is a two hour exam. This is a closed-book, closed-notes and closed-internet exam. Please write all your answers in (one or multiple) blue books.

Unless otherwise stated, you may quote standard theorems from lectures or the textbook, as long as you state them clearly.

1. (20 points) Find the solution to
\[
\begin{align*}
    u'(t) &= \cos \frac{2\pi}{e}(1 + (u(t))^2) \\
    u(0) &= 0,
\end{align*}
\]
where \( u \) is a real-valued function. In addition, determine its maximal interval of existence.

2. (20 points) Consider the initial value problem
\[
\begin{align*}
    u'(t) &= |u(t)|^\alpha \\
    u(0) &= 0,
\end{align*}
\]
where the unknown is a real-valued function \( u(t) \).

   • Show that whenever \( \alpha \in (0, 1) \), the initial value problem has more than one solution \( u : [0, 1] \to \mathbb{R} \).
   • Explain (using any theorems proven in lectures or the textbook) that if \( \alpha \geq 1 \), then the only solution to the initial value problem above is \( u \equiv 0 \).

3. (20 points) Consider the system of ODE
\[
\begin{align*}
    p'(t) &= -q(t) - (q(t))^3 \\
    q'(t) &= p(t).
\end{align*}
\]
Prove that the equilibrium \((0, 0)\) is stable. [Hint: This is a Hamiltonian system.]

4. For each of the following statements, determine whether it is always true. Justify your answers. (10 points each: 2 points for a correct answer, 8 points for a correct justification.)

   (a) Let \( a : \mathbb{R} \to \mathbb{R} \) be a periodic smooth function of period \( T \). Then \( u \equiv 0 \) is the unique smooth function \( u : \mathbb{R} \to \mathbb{R} \) such that both of the following hold:
   
   • \( u \) is periodic of period \( T \).
   • \( u'(t) = a(t)u(t) \).

   (b) Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) such that \( f_n(t) \) converges uniformly to some continuous function \( f : [0, 1] \to \mathbb{R} \) as \( n \to \infty \). Then \( \{f_n\} \) is equicontinuous.
(c) Let \( y(t) \) be a solution to

\[
y''(t) = -(y(t))^3.
\]

Then the maximal interval of existence is \( \mathbb{R} \).

(d) There exists a smooth function \( V : \mathbb{R} \to \mathbb{R} \) such that

\[
y''(t) = -(\nabla V)(y(t))
\]

admits a solution whose maximal interval of existence is not \( \mathbb{R} \).